TEST #1 / STA 5326 (Distribution Theory) / Fall 2000

Directions

- This exam is **closed book** and **closed notes**.
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Please take the exam in the Wilcoxon Room (the departmental library).
- You should remain in the Wilcoxon Room until you finish the test with the following exceptions: You can leave the Wilcoxon Room to use the restrooms or the drinking fountain located in the hallway nearby. You can eat a snack in the hallway outside the Wilcoxon Room. Also, you can come to my office at any time to ask a question or turn in your exam. If you wish to leave the Wilcoxon Room for any other reason, please come by my office and ask permission first.
- **Begin each problem on a new page. Do NOT write on the backs of pages.** Staple the pages together in order when finished.
- Show and explain all your work (including your calculations). **No credit is given without work.** Partial credit is available. (If you know part of a solution – write it down. If you know an approach to a problem, but cannot carry it out – write down this approach.)
- There is no explicit time limit. I am hoping that most people can finish the exam in 3 hours. You must finish by 5:00.
- All the work on the exam should be your own. No "cooperation" is allowed.
- Please remember that people are taking this exam at different times. Do not say anything about the content of the exam to people who have not yet taken it. Please avoid having discussions of the exam questions where others might overhear you.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric series).

Problem 1. Suppose you have a fair coin with the sides labeled $+1$ and -1. Toss this coin 3 times and let X_i be the value observed on the i-th toss. Define $X_4 = X_1 X_2 X_3$. For $i = 1, 2, 3, 4$, define A_i to be the event that $X_i = 1$.

(a) Describe (in general) what must be done to show that four events A, B, C, D are mutually independent.

(b) Show that the events A_1, A_2, A_3, A_4 are *not* mutually independent.

(c) Show that A_1, A_2, A_4 *are* mutually independent.

Problem 2. An urn contains 6 red balls and 14 green balls. Choose 5 balls without replacement from this urn. Let X be the number of red balls drawn. Find the value of $F_X(3.14159)$. (As usual, $F_X(\cdot)$ denotes the cdf of X.)

Problem 3. Let $A_1, A_2, \ldots, A_k$ be any sets. One of DeMorgan's Laws gives another expression for

$$\left(\bigcup_{i=1}^{k} A_i \right)^c .$$

State this one of DeMorgan's Laws and give a proof.

Problem 4. Suppose n people play Russian roulette. Each person has a gun which fires with probability π when the trigger is pulled. (Assume the guns are independent of each other and successive shots of the same gun are independent.) A round of play consists of every one who is still alive raising the guns to their temples and firing simultaneously. Play continues until everyone is dead.

(a) What is the probability that **exactly one** person is still alive after k rounds of play?

[If you cannot do this, then (for somewhat less credit) answer the same question replacing "exactly one" by "at least one".]

(b) The last person (or persons) to die receives a prize (flowers on the grave). What is the probability this prize goes to exactly **two** people?

(Note: The answer may not have a simple form, so do not worry if your answer is messy or contains summations you do not know how to do.)

Problem 5.

(a) Prove that $F(x) = (1 + e^{-x})^{-1}$ is a cdf. (Your answer should contain a careful statement of the conditions that $F(x)$ must satisfy in order to be a cdf.)

(b) Suppose you have software to generate a Uniform(0,1) random variable U. Describe in detail how you would generate a random variable which has the cdf given in part (a).

Problem 6. Let X have pdf $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$. Define the random variable Y by $Y = \mathbf{2}X^{\mathbf{2}}$ if $X \le 0$ and $Y = X^{\mathbf{3}}$ if $X > 0$. Find the range of Y and the (density) pdf of Y.

[If you cannot do this, then (for somewhat less credit) answer the same question using the modified definition $Y = X^2$ if $X \le 0$ and $Y = X^3$ if $X > 0$.]

Problem 7. A children's card game (Barnyard Poker) has a deck of 35 cards. Each card displays one of 7 different animals. There are 5 chickens, 5 cows, 5 pigs, 5 dogs, 5 sheep, 5 horses, and 5 geese.

(a) In this game, players are dealt a "hand" of 5 cards from a well shuffled deck. How many possible **different** 5 card hands are there? (Assume that the ordering of the cards does **not** matter and that all cards with the same animal are considered identical.)

(b) Find the probability that a 5 card hand will contain **no** repeated animals.

Problem 8. A hat contains two coins. Coin #1 has $P(\text{heads}) = 1/4$. Coin #2 has $P(\text{heads}) = 3/4$. A coin is selected at random from the hat and tossed 4 times. (The **same** coin is tossed four times.)

(a) If all 4 tosses are heads, what is the probability that coin #2 has been selected?

(b) If the first 3 tosses are heads, what is the probability that the last toss will be heads?