TEST #2 / STA 5326 (Distribution Theory) / Fall 2000

PLEASE READ: I am still receiving anonymous complaints (now directed to university administrators) about my exam system. In response to these complaints, the format for Test #3 will be different than that for the first two exams. Test #3 will be held in the regular classroom (where lecture is held) on Tuesday, December 12 from 10:00–12:00. You will have **only** two hours for the exam.

Directions

- This exam is **closed book** and **closed notes**.
- You will be given a copy of the Appendix (Table of Common Distributions) from the back of the text. You can use this on any problem except for those where you are explicitly told not to. You do NOT have to prove results from the Appendix.
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Please take the exam in the Wilcoxon Room (the departmental library).
- You should remain in the Wilcoxon Room until you finish the test with the following exceptions: You can leave the Wilcoxon Room to use the restrooms or the drinking fountain located in the hallway nearby. You can eat a snack in the hallway outside the Wilcoxon Room. Also, you can come to my office at any time to ask a question or turn in your exam. If you wish to leave the Wilcoxon Room for any other reason, please come by my office and ask permission first.
- **Begin each problem on a new page. Do NOT write on the backs of pages.** Staple the pages together in order when finished.
- Show and explain all your work (including your calculations). **No credit is given without work.** Partial credit is available. (If you know part of a solution – write it down. If you know an approach to a problem, but cannot carry it out – write down this approach.)
- There is no explicit time limit. I am hoping that most people can finish the exam in 3 hours.
- All the work on the exam should be your own. No "cooperation" is allowed.
- Please remember that people are taking this exam at different times. Do not say anything about the content of the exam to people who have not yet taken it. Please avoid having discussions of the exam questions where others might overhear you.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric series).

Problem 1. Let $M_X(t)$ be the moment generating function of X, and define $S(t) = \log(M_X(t))$. Show that

$$\frac{d}{dt}S(t)\bigg|_{t=0} = EX \quad \text{and} \quad \frac{d^2}{dt^2}S(t)\bigg|_{t=0} = \mathrm{Var}(X)\,.$$

Problem 2. Let $f(x)$ be a pdf and let a be a number such that, if $a \geq x \geq y$ then $f(a) \geq f(x) \geq f(y)$ and, if $a \leq x \leq y$ then $f(a) \geq f(x) \geq f(y)$. Such a pdf is called *unimodal* with a *mode* equal to a. Show that if $f(x)$ is both symmetric and unimodal, then the point of symmetry is a mode.

[Notation: Use c to denote the point of symmetry: $f(c+x) = f(c-x)$ for all x.]

[To receive full credit, you must give a formal argument (a proof) using the definitions of "symmetric" and "unimodal". However, a good informal argument with pictures will be worth substantial partial credit.]

Problem 3. Suppose a distribution has pdf $f(x) = 2x$ for $0 < x < 1$.

(a) Find the median of this distribution.

(b) Let $\mu_n = E[(X - EX)^n]$. Define $\alpha_3 = \mu_3/(\mu_2)^{3/2}$. Find α_3 for this distribution.

Problem 4. Suppose that a rare blood disease strikes men with probability 1.0×10^{-6}, and strikes women with probability 2.0×10^{-6}. In a city of 4 million inhabitants, what is the probability there will be **at least** two people with this disease?

[Assume that men and women are equally numerous. Also, assume that people are independent: whether or not one person gets the disease is independent of what happens to other people.]

Problem 5. An urn contains R red balls and G green balls. Suppose you draw balls **with** replacement until you get your first red ball. Find the mean number of balls drawn.

Problem 6. A fruit wholesaler receives a very large shipment of oranges from a grower. The shipment is considered unacceptable if more than 20.0% of the oranges are sour. (An orange is considered sour if the sugar content falls below a specified level.) The wholesaler plans to randomly sample 100 oranges from the shipment and then determine how many of these oranges are sour.

The wholesaler will accept the shipment if the number of sour oranges in the sample of 100 is less than N. What value of N should they use if they want the probability of accepting an unacceptable shipment to be less than .10? (Use an appropriate approximation.)

[If the above wording confuses you, then for somewhat less credit answer the following: If 30% of the oranges in the shipment are sour, what is the probability that **less than 25** of the oranges in the sample of 100 will be sour? (Use an appropriate approximation.)]

[If you do not know how to carry out the approximation, then (to maximize your partial credit) describe in detail how you would compute the **exact** answers. But do NOT try to actually compute the exact answers!]

Problem 7. Use the appendix as much as possible in the following two parts.

(a) Suppose $X \sim \text{Poisson}(25)$. Find the mgf of $Y = (X - 25)/5$.

(b) Use mgf's to prove the following: If $X_1, X_2, X_3, \ldots, Y$ are random variables with $X_n \sim \text{Poisson}(n)$ and $Y \sim N(0, 1)$, then $(X_n - n)/\sqrt{n} \xrightarrow{d} Y$ (as $n \to \infty$).

Problem 8. Suppose we draw k balls (one by one with**out** replacement) from an urn which contains R red balls and G green balls. Let X be the number of red balls in our sample of k.

(a) Derive a formula for EX. (Do **NOT** use the appendix.)

(b) Derive a formula for $E(X^2)$. (Do **NOT** use the appendix.)

[In this problem, you may use without proof the facts that $P(i\text{-th ball is red})$ is the same for all i, and $P(i\text{-th and } j\text{-th balls are red})$ is the same for all $i \neq j$.]