**TEST #2**
**STA 5326** **Name:**__________________________________
**October 30, 2003**

**Please read the following directions.**
**DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO**

# Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Show and explain your work (including your calculations). **No credit is given without work.** But don't get carried away! Show just enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages and a total of **100 points**.

**Problem 1.** Cars pass a certain spot according to a Poisson process with rate $\lambda = 0.40$ cars per minute.

**(a)** (12 pt) How long should you wait in order to have a probability of about .80 of seeing at least two cars? (Calculate this waiting time accurately enough so that the probability lies between 0.78 and 0.82.)

[**Problem 1 continued** ]

**(b)** (14 pt)If you wait for 1000 minutes, what is the approximate probability that the number of cars you see will be greater than 430?

**Problem 2.** Let $X$ have the (textbook's) negative binomial distribution with pmf

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, \quad x = 0, 1, 2, \ldots$$

where $0 < p < 1$ and $r > 0$ is an integer.

**(a)** (14 pt) Calculate the mgf of $X$. (Do this directly from the definition of the mgf.)

**(b)** (12 pt) Define a new random variable by $Y = 2pX$. Use mgf's to show that, as $p \downarrow 0$, the distribution of $Y$ converges to that of a chi-squared random variable with $2r$ degrees of freedom. (Look up the necessary mgf's in the appendix.)

**Problem 3.** Let $X$ have pdf $f(x)$.

**(a)** (10 pt) We say that $f(x)$ is symmetric about the point $a$ if $f(a+z) = f(a-z)$ for all $z > 0$. Show that if $X$ is symmetric about $a$ and $EX$ exists, then $EX = a$.

**(b)** (10 pt) We say that $f(x)$ is unimodal with mode $b$ if $f(b) \geq f(x) \geq f(y)$ whenever $b \leq x \leq y$ or $b \geq x \geq y$. Show that if $f(x)$ is symmetric about $a$ and unimodal with mode $b$, then $a = b$. (To simplify matters, you may assume the mode is unique which guarantees that $f(b) > f(x)$ for $b \neq x$.)

**Problem 4.** (12 pt) Show that

$$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z}\, dz = \sum_{y=0}^{\alpha-1} \frac{x^y e^{-x}}{y!}, \quad \alpha = 1, 2, 3, \ldots$$

(You may prove this either by doing a calculation or by using facts about a Poisson process. Hint for latter approach: Consider a Poisson process with rate $\lambda = 1$ at time $x$.)

**Problem 5.** (16 pt) Suppose the flow of traffic at a street corner is a sequence of Bernoulli trials with the probability of a car passing during any given second equal to $p$. Suppose also that a pedestrian requires three seconds without cars in order to safely cross the street. Eight blind pedestrians cross the street one by one, departing at one second intervals. (That is, if we start time from zero, the blind pedestrians begin crossing at times 0, 1, 2, 3, 4, 5, 6, 7.) Let $X$ be the number who get safely across. Find the mean and variance of $X$.