TEST #1
STA 5326
September 28, 2006

Name: Solution

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** pages and a total of **100 points**.

Problem 1. (14 pt) Suppose $Y = 7X + 3$ where X has density (pdf) $f_X(x) = 4e^{-4x}$ for $x > 0$. Find the **cumulative distribution function (cdf)** of Y.
(Make sure to define the cdf for all values of y.)

Solution 1.

$$F_Y(y) = P(Y \le y) = P(7X+3 \le y)$$
$$= P(X \le \frac{y-3}{7})$$
$$= P(0 < x \le \frac{y-3}{7})$$
$$= \int_0^{\frac{y-3}{7}} 4 \cdot e^{-4x} dx$$
$$= 1 - e^{\frac{4(3-y)}{7}} \quad \text{for } x > 0.$$

$$x > 0 \Rightarrow Y = 7X + 3 > 3.$$

$$\Rightarrow F_Y(y) = \begin{cases} 1 - e^{\frac{4(3-y)}{7}} & y > 3 \\ 0 & \text{o.w.} \end{cases}$$

Solution 2.

$Y = 7X+3$ is monotone increasing.

$$F_Y(y) = F_X(g^{-1}(y)) \qquad g^{-1}(y) = \frac{y-3}{7}$$

$$F_X(x) = \int_0^x 4e^{-4z} dz$$
$$= -e^{-4z} \Big|_0^x$$
$$= 1 - e^{-4x} \qquad x > 0$$

$$F_Y(y) = F_X(g^{-1}(y)) = 1 - e^{-\frac{4}{7}(y-3)} \quad (x > 0)$$

$$x > 0 \Rightarrow y > 3$$

$$\Rightarrow F_Y(y) = \begin{cases} 1 - e^{-\frac{4}{7}(y-3)} & y > 3 \\ 0 & \text{o.w.} \end{cases}$$

Problem 2. (14 pt) A colorblind man draws balls **WITH** replacement from an urn containing 7 red balls, 3 orange balls, and 4 white balls, announcing the color of each ball after drawing it. Unfortunately, he mistakes a red ball for an orange ball (announcing "orange" when he should announce "red") with probability 1/4, and mistakes an orange ball for a red ball (announcing "red" when he should announce "orange") with probability 1/10. He never makes mistakes with the white balls. If the man draws two balls and announces "red, red", what is the probability that this is what he actually drew?

Solution 1. Let dR = ~~the~~ a Red ball is picked $\quad P(dR) = \frac{7}{14} = \frac{1}{2}$

aR = a Red ball is announced.

dO = an orange ball is picked. $\quad P(dO) = \frac{3}{14}$

Then, the question is ask for $P(dRdR/aRaR)$

Since the balls are drawn with replacement, the 1st pick and the second one are independent.

$$P(dRdR/aRaR) = [P(dR/aR)]^2$$

$$P(dR/aR) = \frac{P(dR \cap aR)}{P(aR)} = \frac{P(aR/dR) \cdot P(dR)}{P(aR)}$$

$$P(aR/dR) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(dR) = \frac{1}{2}$$

$$P(aR) = P(aRdR) + P(aRdO) + P(aRdW)$$

$$= P(aR/dR) \cdot P(dR) + P(aR/dO) \cdot P(dO) + P(aR/dW) \cdot P(dW)$$

$$= \frac{3}{4} \cdot \frac{1}{2} + \frac{1}{10} \cdot \frac{3}{14} + 0 \cdot \frac{4}{14}$$

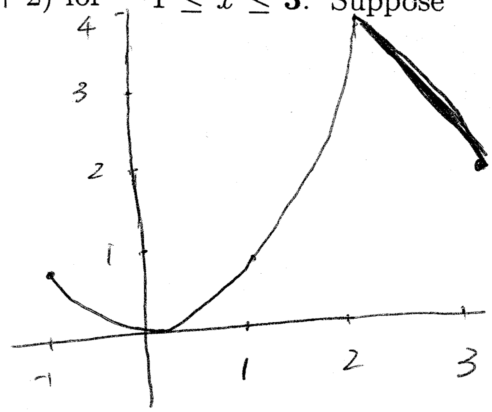
$$P(dR/aR) = \frac{\frac{3}{4} \times \frac{1}{2}}{\frac{3}{4} \times \frac{1}{2} + \frac{3}{140}} = \left(\frac{\frac{3}{8}}{\frac{3}{8} + \frac{3}{140}} \right)^2 =$$

$$\Rightarrow P(dRdR/aRaR) = \left(\frac{140}{148} \right)^2 = \left(\frac{35}{37} \right)^2$$

Problem 3. (14 pt) Let X have density (pdf) $f_X(x) = \frac{1}{12}(x+2)$ for $-1 \le x \le 3$. Suppose $Y = g(X)$ where

$$g(x) = \begin{cases} x^2 & \text{for } -1 \le x \le 2, \\ 8-2x & \text{for } 2 < x \le 3. \end{cases}$$

Find the density (pdf) of Y.



$$g^{-1}(y) = \begin{cases} \pm\sqrt{y} & 0 \le y \le 4 \quad -1 \le x \le 2 \\ \frac{8-y}{2} & 2 \le y \le 4 \quad 2 < x \le 3 \end{cases}$$

$$\Rightarrow \quad g^{-1}(y) = \begin{cases} -\sqrt{y} & -1 \le x \le 0 \quad 0 \le y \le 1 \quad A_1 \\ \sqrt{y} & 0 \le x \le 2 \quad 0 \le y \le 4 \quad A_2 \\ \frac{8-y}{2} & 2 < x \le 3 \quad 2 \le y \le 4 \quad A_3 \end{cases}$$

all are monotone in the domain.

$$f_Y(y) = \sum_{i=1}^{3} f_X(g^{-1}(y)) \left| \frac{d g^{-1}(y)}{dy} \right| \cdot I_{A_i}(y)$$

$$= \frac{1}{12}(-\sqrt{y}+2) \cdot \left| -\frac{1}{2} \frac{1}{\sqrt{y}} \right| \cdot I_{(0,1)}(y) + \frac{1}{12}(\sqrt{y}+2) \left| \frac{1}{2} \cdot \frac{1}{\sqrt{y}} \right| \cdot I_{(0,4)}(y)$$

$$+ \frac{1}{12}\left(\frac{8-y}{2}+2\right) \cdot \left| -\frac{1}{2} \right| \cdot I_{(2,4)}(y)$$

So
$$f_Y(y) = \begin{cases} \frac{1}{12}(-\sqrt{y}+2) \cdot \frac{1}{2\sqrt{y}} + \frac{1}{12}(\sqrt{y}+2) \cdot \frac{1}{2\sqrt{y}} & \text{if } 0 < y \le 1 \\ \frac{1}{12}(\sqrt{y}+2) \cdot \frac{1}{2\sqrt{y}} & \text{if } 1 < y < 2 \\ \frac{1}{12}(\sqrt{y}+2) \cdot \frac{1}{2\sqrt{y}} + \frac{1}{24}\left(\frac{12-y}{2}\right) & \text{if } 2 < y \le 4 \\ 0 & \text{o.w.} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{6}{\sqrt{y}} & 0 < y \le 1 \\ \frac{1}{24}\left(1+\frac{2}{\sqrt{y}}\right) & 1 < y < 2 \\ \frac{1}{24}\left[\left(1+\frac{2}{\sqrt{y}}\right)+\left(6-\frac{y}{2}\right)\right] & 2 \le y \le 4 \\ 0 & \text{o.w.} \end{cases}$$

Problem 4. (14 pt) A man has bought a small bottle of 20 aspirin tablets. Unknown to him, terrorists have replaced 8 of the pills with a similar looking popular laxative. A dose of two of these pills is sufficient to ensure catastrophic results, whereas one pill causes only mild discomfort. While suffering from a bad headache, the man consumes four tablets chosen at random from the bottle. What is the probability he will suffer catastrophic results?

Solution 1. $P(\text{Suffer Catastrophic results}) = P(\#\text{ of bad pills} \geq 2)$

$$= P(N \geq 2)$$

$$P(N\geq 2) = P(N=2) + P(N=3) + P(N=4)$$

$$P(N=2) = \frac{\binom{8}{2}\binom{12}{2}}{\binom{20}{4}}$$

$$P(N=3) = \frac{\binom{8}{3}\binom{12}{1}}{\binom{20}{4}}$$

$$P(N=4) = \frac{\binom{8}{4}}{\binom{20}{4}}$$

Solution 2:

$$P(N\geq 2) = 1 - P(N<2)$$

$$= 1 - P(N=0) - P(N=1)$$

$$P(N=0) = \frac{\binom{12}{4}}{\binom{20}{4}}$$

$$P(N=1) = \frac{\binom{12}{3}\binom{8}{1}}{\binom{20}{4}}$$

Problem 5. (14 pt) A game show always proceeds in the following manner: A prize (a new car) is placed at random behind either door $\boldsymbol{A}$, $\boldsymbol{B}$, or $\boldsymbol{C}$. Behind each of the other two doors there is placed a goat. The contestant selects a door. Without revealing to the contestant what is behind her selected door, one of the other doors is opened to reveal a goat. If there is a goat behind both of the other doors, the door that is opened is chosen at random. In today's game, the contestant chooses door $\boldsymbol{A}$ and then door $\boldsymbol{B}$ is opened to reveal a goat. Given this, what is the probability the prize is behind door $\boldsymbol{C}$?

Note: In this problem "at random" always means that all the possibilities are equally likely.

Solution 1 $P(C \text{ has prize} \mid B \text{ is open})$ assume A is selected deterministically

$$= \frac{P(C \text{ has prize. \& } B \text{ is open})}{P(B \text{ is open})}$$

$$= \frac{P(B \text{ is open} \mid C \text{ has prize}) \cdot P(C \text{ has prize})}{P(B \text{ is open})}$$

$P(C \text{ has prize}) = \frac{1}{3}$ since the prize is equally likely to be behind door A. B and C.

$P(B \text{ is open} \mid C \text{ has prize}) = 1$ since only the door with goat will be open.

$$P(B \text{ is open}) = P(B \text{ is open. } C \text{ has the prize}) + P(B \text{ is open. } A \text{ has the prize}) + P(B \text{ is open. } B \text{ has the prize})$$

$P(B \text{ is open. } c \text{ has the prize}) = \frac{1}{3} \cdot 1$

$P(B \text{ is open. } A \text{ has the prize}) = P(B \text{ is open} \mid A \text{ has prize}) \cdot P(A \text{ has the prize}) = \frac{1}{2} \cdot \frac{1}{3}$

$P(B \text{ is open. } B \text{ has the prize}) = 0$

$$\Rightarrow P(C \text{ has prize} \mid B \text{ is open}) = \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{2} \cdot \frac{1}{3}} = \frac{2}{3}.$$

~~Solution 2: $P(A) + P(B) + P(C) = 1$~~

Problem 6. (14 pt) Suppose X has pdf defined by $f(x) = cx$ for $\mathbf{2} < x < \mathbf{3}$ and $f(x) = 0$ otherwise.

(a) What is the value of the constant c?

$$\int_2^3 f(x)\,dx = 1$$

$$\Rightarrow \int_2^3 cx \; dx = 1$$

$$\Rightarrow \frac{c}{2} x^2 \Big|_2^3 = 1$$

$$\Rightarrow \frac{c}{2}(9-4) = 1$$

$$c = \frac{2}{5}$$

(b) Calculate $E(\log X)$.
(If you could not do part (a), just leave the constant as c in your answer.)

$$E \log X = \int_2^3 \log X \cdot \frac{2}{5} \cdot X \; dx$$

$$= \frac{1}{5} \int_2^3 \log X \cdot 2x \, dx = \frac{1}{5} \int_2^3 \log x \cdot dx^2$$

Let $V = \log X$. $dV = \frac{1}{x} dx$

$u = x^2 \quad du = 2x\,dx$

$$E \log X = \frac{1}{5}\left[\log x \cdot x^2 \Big|_2^3 - \int_2^3 x^2 \, d\log x\right]$$

$$= \frac{1}{5}\left[\log x \cdot x^2 \Big|_2^3 - \int_2^3 x^2 \cdot \frac{1}{x} dx\right]$$

$$= \frac{1}{5}\left[\log x \cdot x^2 - \frac{1}{2} x^2\right]\Big|_2^3$$

$$= \frac{1}{5}\left[9 \log 3 - 4 \log 2 - \frac{1}{2}(9-4)\right]$$

$$= \frac{9}{5} \log 3 - \frac{4}{5} \log 2 - \frac{1}{2}$$

No work is required for the problems on this page. Just give the answers.

Problem 7. (6 pt) Let A, B, C, D, E be 5 real numbers. Keeping in mind that multiplication is commutative so that order does not matter, using only these numbers, how many different products of 11 numbers can you form? (Some examples are $A^2B^2C^2DE^4$, B^3E^8, C^{11}, etc.)

Here. problem is similar to Exercise 1.19.

$n=5$. $r=11$

the total # of diff. products is $\binom{n+r-1}{r}$

$= \binom{15}{11}$

Problem 8. (5 pt) State a formula for

$$P(A \cap B^c \cap C)$$

in terms of $P(A), P(B), P(C), P(A \cap B), P(A \cap C), P(B \cap C), P(A \cap B \cap C)$.
(You don't need to use all of them in your formula.)

$P(A \cap B^c \cap C)$

$= P(A \cap C) - P(A \cap B \cap C)$

Problem 9. (5 pt) Suppose $P(A_1) = P(A_2) = \cdots = P(A_6) = P(A_7) = \beta$. Which one of the following must **always** be true? (Circle the correct choice.)

a) $P(A_1 \cap \cdots \cap A_7) \leq 7\beta$ b) $P(A_1 \cap \cdots \cap A_7) \geq 7\beta$ c) $P(A_1 \cap \cdots \cap A_7) \leq \beta^7$

d) $P(A_1 \cap \cdots \cap A_7) \geq \beta^7$ e) $P(A_1^c \cap \cdots \cap A_7^c) \geq (1-\beta)^7$ f) $P(A_1 \cup \cdots \cup A_7) \leq 7\beta$

g) $P(A_1 \cup \cdots \cup A_7) \geq 7\beta$ h) $P(A_1 \cup \cdots \cup A_7) \leq \beta^7$ i) $P(A_1 \cup \cdots \cup A_7) \geq \beta^7$