

Please read the following directions.

**DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO**

## Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the “Table of Common Distributions” given in the back of the text.)
- Show and explain your work (including your calculations) on all the problems. **No credit is given without work**. But don’t get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No “cooperation” is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- If you need more room to work a problem, use the back of the same page and write “work on back” to indicate that you have done so.
- The exam has **9** pages and a total of **100 points**.

**Problem 1.** A random right triangle is constructed in the following manner. Let  $d > 0$  be a fixed constant. Let  $X$  be a random angle with density (pdf) given by

$$f_X(x) = \frac{4 \tan(x)}{\pi(1 + \tan(x))} \quad \text{for } 0 < x < \frac{\pi}{2}$$

and construct a triangle with two vertices at the points  $(0, 0)$  and  $(d, 0)$  having angle  $X$  at  $(0, 0)$  and a right angle at  $(d, 0)$ . Let  $Y$  be the height of this random triangle.

Note:  $\frac{d}{du} \tan^{-1}(u) = \frac{1}{1 + u^2}$  where  $\tan^{-1}$  denotes the inverse of the  $\tan$  function.

(a) (8 pt) Find the density (pdf) of  $Y$ .

(b) (4 pt) Find  $EY$ .

**Problem 2.** Cars pass a certain spot according to a Poisson process with rate  $\lambda = 0.50$  cars per minute. Suppose Al watches from this spot for 800 minutes on Monday, and Bob watches for 800 minutes on Tuesday.

(a) (6 pt) Let  $X$  be the total number of cars seen by Al and Bob (their combined total). What is the distribution of  $X$ ?

(Give the name of the distribution and the values of any parameters.)

(b) (9 pt) What is the probability that Al and Bob observe exactly the same number of cars? (Use an appropriate approximation.)

**Problem 3.**

**(a)** (5 pt) Does there exist a distribution for which  $M_X(t) = t/(1+t)$  for  $|t| < 1$ ? Answer “Yes” or “No”. If yes, find it. If no, prove it.

**(b)** (4 pt) Does there exist a distribution for which  $M_X(t) = 1 - |t|$  for  $|t| < 1$ ? Answer “Yes” or “No”. If yes, find it. If no, prove it.

**Problem 4.** Let  $X_1, X_2, \dots, X_7$  be iid random variables with density  $f(x) = 3x^2$  for  $0 < x < 1$ . Define  $W = \min X_i$ .

(a) (8 pt) Find the cdf of  $W$ .

(b) (6 pt) Calculate  $EW$ .

Given: A pdf  $f(x)$  is said to be symmetric about the value  $a$  if  $f(a + u) = f(a - u)$  for all  $u$ . If the pdf of  $X$  is symmetric about  $a$  and  $EX$  exists, then  $EX = a$ .

**Problem 5.** (10 pt) The skewness of a random variable  $X$  is defined to be

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} \quad \text{where } \mu_k = E[(X - EX)^k].$$

Show that, if the pdf of  $X$  is symmetric about  $a$ , then  $\alpha_3 = 0$ .

(Assume that  $\mu_3$  and  $\mu_2$  are finite. You may use the “given” information without proof.)

**Problem 6.** (12 pt) Let  $X$  and  $Y$  be independent random variables with  $X \sim N(\mu = 5, \sigma^2 = 7)$  and  $Y \sim \text{Exp}(\beta = 11)$ . Find the moment generating function (mgf) of

$$Z = 2X - 3Y + 47.$$

Make sure to specify the range of values where the mgf is finite (well-defined).  
(Use the appendix to find any mgf's you need.)

**Problem 7.** Suppose a coin (with probability  $p$  of heads) is tossed repeatedly. Let  $\alpha$  and  $\beta$  be fixed positive integers. Define  $X$  to be the number of heads in the first  $\alpha$  tosses, and  $Y$  to be the number of **tails** observed before obtaining the  $\beta$ th **head**.

(a) (4 pt) What is the distribution of  $X$ ? (State the name of the distribution and the values of any parameters.)

(b) (4 pt) What is the distribution of  $Y$ ? (State the name of the distribution and the values of any parameters.)

(c) (6 pt) Let  $F_X$  and  $F_Y$  denote the cdf's of these two random variables. State a relationship between these two cdf's. (Give a formula relating them.) Prove this relationship.

**Problem 8.** A store has 50 boxes of the breakfast cereal “Sugar Munchies” and 10 of these contain a prize.

(a) (9) How many boxes must you buy in order to have a probability of at least .90 of winning a prize?

(b) (5) If you buy 7 boxes, what is the **variance** of the number of prizes you get?