TEST #2
STA 5326
November 2, 2006

Name: Solution

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- If you need more room to work a problem, use the back of the same page and write "work on back" to indicate that you have done so.
- The exam has **9** pages and a total of **100 points**.

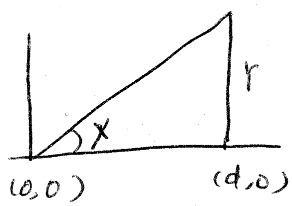
Problem 1. A random right triangle is constructed in the following manner. Let $d > 0$ be a fixed constant. Let X be a random angle with density (pdf) given by

$$f_X(x) = \frac{4\tan(x)}{\pi(1+\tan(x))} \quad \text{for } 0 < x < \frac{\pi}{2}$$

and construct a triangle with two vertices at the points $(0,0)$ and $(d,0)$ having angle X at $(0,0)$ and a right angle at $(d,0)$. Let Y be the height of this random triangle.

Note: $\frac{d}{du}\tan^{-1}(u) = \frac{1}{1+u^2}$ where $\tan^{-1}$ denotes the inverse of the tan function.

(a) (8 pt) Find the density (pdf) of Y.



$$\frac{Y}{d} = \mathrm{tg}\, X \qquad 0 < X < \frac{\pi}{2}$$

$$\Rightarrow Y = d\,\mathrm{tg}\, X \quad \text{and} \quad Y > 0$$

$$\Rightarrow g^{-1}(y) = \mathrm{arctg}\left(\frac{y}{d}\right) \qquad 0 < y < +\infty$$

$$\text{So } f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{dg^{-1}(y)}{dy}\right|$$

$$= \frac{4 \cdot y/d}{\pi(1+y/d)} \cdot \frac{\frac{1}{d}}{1+y^2/d^2}$$

$$= \frac{4yd}{\pi(y+d)(d^2+y^2)} \quad (0 < y < +\infty)$$

$$\frac{dg^{-1}(y)}{dy} = d\left(\mathrm{arctg}\left(\frac{y}{d}\right)\right)$$

$$= \frac{1}{1+\left(\frac{y}{d}\right)^2} \cdot \frac{d\left(\frac{y}{d}\right)}{dy}$$

$$= \frac{1}{1+\frac{y^2}{d^2}} \cdot \frac{1}{d}$$

(b) (4 pt) Find EY.

$$EY = \int_0^{+\infty} \frac{y \cdot 4yd}{\pi(d+y)(d^2+y^2)}\, dy$$

$$= \int_0^{+\infty} 4d \cdot \frac{y^2}{(d+y)(d^2+y^2)}\, dy$$

$$\lim_{y\to\infty} \frac{y^2}{d^2+y^2} = 1 \quad \Rightarrow \quad \frac{y^2}{(d+y)(d^2+y^2)} \approx \frac{1}{y} \text{ for large } y.$$

$$\Rightarrow EY = \infty \quad \text{because} \quad \int_c^{\infty} \frac{dx}{x} = \infty \quad \text{for } c > 0$$

Problem 2. Cars pass a certain spot according to a Poisson process with rate $\lambda = 0.50$ cars per minute. Suppose Al watches from this spot for 800 minutes on Monday, and Bob watches for 800 minutes on Tuesday.

(a) (6 pt) Let X be the total number of cars seen by Al and Bob (their combined total). What is the distribution of X?
(Give the name of the distribution and the values of any parameters.)

Al : $X_1 \sim$ Poisson (800λ)
Bob : $X_2 \sim$ poisson (800λ)
$X_1 \cdot X_2$ are indep. $\Rightarrow X_1 + X_2 \sim$ Poisson $(800\lambda + 800\lambda)$.
$=$ poisson (800)

(b) (9 pt) What is the probability that Al and Bob observe exactly the same number of cars? (Use an appropriate approximation.)

Because 800 is a large #.
$X_1 \; X_2 \overset{app.}{\sim} N(400, 400)$ and they are indep.
$\Rightarrow X_1 - X_2 \overset{app.}{\sim} N(0, 800)$

P(Al and Bob observe exactly the same # of cars)

$= P(|X_1 - X_2| < \frac{1}{2})$

$= P\left(\frac{|X_1 - X_2\} - 0|}{\sqrt{800}} < \frac{1}{2\sqrt{800}}\right)$

$= P\left(|z| < \frac{1}{2\sqrt{800}}\right) \quad z \sim N(0,1)$

$\approx \frac{1}{\sqrt{2\pi}} \cdot 2 \cdot \frac{1}{2\sqrt{800}}$

$= \frac{1}{\sqrt{2\pi \cdot 800}} = \frac{1}{40\sqrt{\pi}}$.

Problem 3.

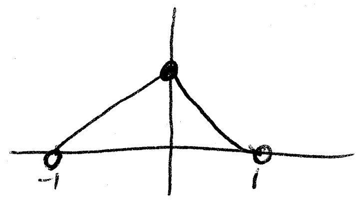
(a) (5 pt) Does there exist a distribution for which $M_X(t) = t/(1+t)$ for $|t| < 1$? Answer "Yes" or "No". If yes, find it. If no, prove it.

No.

because $M_X(0) = \frac{0}{1+0} = 0$. while a mgf of any distributis should be 1 when $t=0$.

(b) (4 pt) Does there exist a distribution for which $M_X(t) = 1 - |t|$ for $|t| < 1$? Answer "Yes" or "No". If yes, find it. If no, prove it.

No.



An mgf must be convex and smooth, which mean infinitely differentiable) in the range where it is finite.

Here. $M_X(t)$ is not differentiable at $t=0$.

Problem 4. Let $X_1, X_2, \ldots, X_7$ be i.i.d. random variables with density $f(x) = 3x^2$ for $0 < x < 1$. Define $W = \min X_i$.

(a) (8 pt) Find the cdf of W.

$$P(W > w) = P(\min X_i > w) \qquad w > 0$$
$$= P(X_1 > w, X_2 > w, \cdots X_7 > w)$$
$$= \prod_{i=1}^{7} P(X_i > w) \quad \text{by independence}$$
$$= \prod_{i=1}^{7} (1 - P(X_i \le w))$$
$$= (1 - P(X \le w))^7 \quad \text{because of identical dist.}$$
$$= (1 - F_X(w))^7$$

$$F_W(w) = P(W \le w) = 1 - P(W > w) = 1 - (1 - F_X(w))^7$$

$$F_X(w) = \int_0^w 3x^2 dx = \int_0^w dx^3 = x^3 \Big|_0^w = w^3$$

$$\Rightarrow \quad F_W(w) = [1 - (1 - w^3)^7] \qquad 0 < w < 1$$

(b) (6 pt) Calculate EW.

$$EW = \int_0^1 w \cdot f_W(w)\, dw$$
$$= \int_0^1 \cdot w \cdot (-7) \cdot (1-w^3)^6 \cdot (-3w^2)\, dw$$
$$= 28 \int_0^1 w^3 (1-w^3)^6 dw. \qquad \text{Let } v = w^3, \ w = v^{1/3}$$
$$= 28 \int_0^1 v (1-v)^6 \cdot \frac{1}{3} \cdot v^{-2/3} dv$$
$$= 28 \int_0^1 \frac{1}{3} \cdot v^{\frac{4}{3}-1} \cdot (1-v)^6 dv$$
$$= 7 \cdot \int_0^1 v^{\frac{4}{3}-1} (1-v)^{7-1} dv$$
$$= 7 \cdot B(\tfrac{4}{3}, 7) = 7 \cdot \frac{\Gamma(\frac{4}{3}) \cdot 6!}{\Gamma(\frac{4}{3} + 7)} = \frac{7! \ \Gamma(\frac{4}{3})}{\prod_{i=0}^{6} (\frac{4}{3} + i) \cdot \Gamma(\frac{4}{3})}$$
$$= \frac{7!}{\prod_{i=1}^{6} (\frac{4}{3} + i)}$$
$$\approx .453$$

Given: A pdf $f(x)$ is said to be symmetric about the value a if $f(a+u) = f(a-u)$ for all u. If the pdf of X is symmetric about a and EX exists, then $EX = a$.

Problem 5. (10 pt) The skewness of a random variable X is defined to be

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} \quad \text{where } \mu_k = E[(X - EX)^k].$$

Show that, if the pdf of X is symmetric about a, then $\alpha_3 = 0$.
(Assume that μ_3 and μ_2 are finite. You may use the "given" information without proof.)

pdf $f(x)$ is symmetric about a. $\mu_1 = a$.

$$\mu_2 = \int_{-\infty}^{\infty} (x-a)^2 f(x)\, dx$$

≥ 0. $\mu_2 = 0$. only if $(x-a)^2 = 0$ $\Rightarrow x = a$. $\Rightarrow f(x)$ doesn't exist

So. $\mu_2 > 0$

$$\mu_3 = \int_{-\infty}^{\infty} (x-a)^3 f(x)\, dx$$

$$= \int_a^{\infty} (x-a)^3 f(x)\, dx + \int_{-\infty}^{a} (x-a)^3 f(x)\, dx$$

$$= \underbrace{\int_0^{\infty} u^3 f(u+a)\, du}_{x-a=u} + \underbrace{\int_{\infty}^{0} (-u)^3 f(a-u)(-du)}_{x-a=-u}$$

$$= \int_0^{\infty} u^3 f(u+a)\, du + \int_{\infty}^{0} u^3 f(a-u)\, du$$

$$= \int_0^{\infty} u^3 f(u+a)\, du - \int_0^{\infty} u^3 f(a+u)\, du$$

because $f(a-u) = f(a+u)$ $\forall u$

$= 0$

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{0}{(\mu_2)^{3/2}} = 0$$

Problem 6. (12 pt) Let X and Y be independent random variables with $X \sim N(\mu = 5, \sigma^2 = 7)$ and $Y \sim \text{Exp}(\beta = 11)$. Find the moment generating function (mgf) of

$$Z = 2X - 3Y + 47\,.$$

Make sure to specify the range of values where the mgf is finite (well-defined).
(Use the appendix to find any mgf's you need.)

$X \cdot Y$ are independent.

$$M_Z(t) = E(e^{Zt})$$
$$= E(e^{(2X-3Y+47)t})$$
$$= E(e^{2X}) \cdot E(e^{-3Y}) \cdot e^{47t}$$
$$= M_X(2t) \cdot M_Y(-3t) \cdot e^{47t}$$

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \qquad \mu = 5 \,.\quad \sigma^2 = 7$$
$$= e^{5t + \frac{7}{2}t^2}$$
$$M_X(2t) = e^{10t + \frac{7}{2}\cdot 4t^2} = e^{10t + 14t^2}$$

$$M_Y(t) = \frac{1}{1-\beta t} = \frac{1}{1-11t} \qquad \beta = 11$$
$$M_Y(-3t) = \frac{1}{1+33t}$$

$$M_Z(t) = e^{10t+14t^2} \cdot \frac{1}{1+33t} \cdot e^{47t}$$
$$= \frac{e^{57t+14t^2}}{1+33t}$$

$$M_Z(t) > 0 \quad \Rightarrow \quad 1+33t > 0 \quad \Rightarrow \quad t > \frac{-1}{33}$$

Problem 7. Suppose a coin (with probability p of heads) is tossed repeatedly. Let α and β be fixed positive integers. Define X to be the number of heads in the first α tosses, and Y to be the number of **tails** observed before obtaining the βth **head**.

(a) (4 pt) What is the distribution of X? (State the name of the distribution and the values of any parameters.)

$X \sim$ Binomial (α, p)

(b) (4 pt) What is the distribution of Y? (State the name of the distribution and the values of any parameters.)

$Y \sim$ Negative Binomial (β, p)

(c) (6 pt) Let F_X and F_Y denote the cdf's of these two random variables. State a relationship between these two cdf's. (Give a formula relating them.) Prove this relationship.

P(# of heads in the first α tosses $< \beta$)

= P(# of tails before the βth head $\geq \alpha - \beta$)

$$\Rightarrow P(X < \beta) = P(Y \geq \alpha + \beta)$$

$$\Rightarrow P(X \leq \beta - 1) = P(Y \geq \alpha + \beta)$$

$$F_X(\beta - 1) = 1 - P(Y \leq \alpha + \beta)$$

$$= 1 - F_Y(\alpha + \beta)$$

So. the relationship b/t these two cdf's is

$$\boxed{F_X(\beta - 1) = 1 - F_Y(\alpha + \beta)}$$

or $\quad 1 - F_X(\beta) = F_Y(\alpha + \beta - 1)$

Problem 8. A store has 50 boxes of the breakfast cereal "Sugar Munchies" and 10 of these contain a prize.

(a) (9) How many boxes must you buy in order to have a probability of at least .90 of winning a prize?

P(winning a prize) = P(at least one box containing a prize)

Let X be the # of boxes bought.

P(winning a prize / X) = P(at least 1 box with a prize / X)

= 1 − P(no box with a prize / X)

≥ .9

⇒ P(no box with a prize / X) < .1

$$\Rightarrow \frac{\binom{40}{X}}{\binom{50}{X}} < .1$$

by error and trial $X \geq 10$.

(b) (5) If you buy 7 boxes, what is the **variance** of the number of prizes you get?

X = # of prizes you get

This is a hypergeometric distribution

$$Var(X) = \frac{K \cdot M}{N} \cdot \frac{N-M}{N} \cdot \frac{N-K}{N-1}$$

$N = 50$
$M = 10.$
$K = 7$

$$= \frac{7 \times 10 \cdot 40 \cdot 43}{50 \cdot 50 \cdot 49}$$

$= .983$