TEST #1
STA 5326 **Name:**____________________________________
September 27, 2007

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The different parts of a problem are often unrelated.
- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.

Problem 1. A monkey types **8 digits** at random. (Each keystroke is independent of the others with all of the **10** possibilities `0,1,2,3,4,5,6,7,8,9` being equally likely.

(a) (13%) Let X_i denote the number of digits that the monkey types exactly i times. For example, if the monkey types `43266999` then $X_1 = 3$, $X_2 = 1$, $X_3 = 1$ and $X_i = 0$ for $i \geq 4$.

Find the probability distribution of $X_{\mathbf{4}}$.

[**Problem 1 continued**]

(b) (7%) What is the probability the monkey types at least two `9`'s?

(c) (7%) What is the conditional probability the monkey types at least two `9`'s, given that it types at least one `9`?

(d) (3%) Consider the events:

$A = \{$the monkey types only odd digits$\}$ and $B = \{$the monkey types only even digits$\}$.

These are events are ... (Circle the correct completion.)

a) independent **b**) mutually exclusive **c**) both (a) and (b) **d**) neither (a) nor (b)

Problem 2. (14%) Suppose X has density (pdf) $f_X(x) = e^{-x}$ for $x > 0$.

Find the density (pdf) of $Y = e^X$. (Make sure to specify where it is positive.)

Problem 3. Suppose X has mass function (pmf) $f_X(x) = (e-1)e^{-x}$ for $x = 1, 2, 3, \ldots$

(a) (10%) Find the mass function (pmf) of $Y = e^X$.

(b) (3%) Let P_X denote the induced probability function of X.
What is the value of $\boldsymbol{P_X([1, 2) \cup (4, 5])}$?

In this problem we abbreviate events by omitting the intersection symbol $\cap$ so that $A \cap B = AB$, $A \cap B \cap C = ABC$, etc.

Problem 4. (10%) Let A, B, C, D be any events. (We are NOT assuming any special properties like independence.) Find an expression for $P(AB \cup AC \cup AD \cup BC)$ as a sum of probabilities of various intersections of these events. (Simplify your answer as much as possible.)

Problem 5. Consider the "Gambler's Ruin" problem with a biased coin having probability p of Heads (win a dollar) and $1-p$ of Tails (lose a dollar). Let $\psi(z)$ denote the probability of reaching a given goal of g dollars starting with an initial fortune of z dollars.

(a) (10%) Use the Law of Total Probability to find an equation that $\psi(z)$ must satisfy. (Show the argument in detail.)

You are given that $\psi(z) = \dfrac{R^z - 1}{R^g - 1}$ where $R = (1-p)/p$. Using this, answer the following.

(b) (6%) What is the probability the gambler's first two tosses were heads, given that he reached the goal?

Problem 6. Suppose $X_1, X_2, X_3, \ldots$ are independent Poisson(1) random variables. Define

$$Y = \sum_{i=1}^{\infty} \frac{X_i}{2^i} = \frac{X_1}{2} + \frac{X_2}{4} + \frac{X_3}{8} + \cdots.$$

Calculate the following.

(a) (7%) $E(Y)$

(b) (6%) $\mathrm{Var}(Y)$

If you have finished everything else and have checked your answers, then try this:

Problem 7. (4%) Suppose X has density (pdf) $f_X(x) = e^{-x}$ for $x > 0$. Let $Y = \frac{\sin(X)}{X}$. Find $f_Y(0)$, the density of Y at zero.