TEST #1 *(with added remarks)*
STA 5326
September 27, 2007

Name: Key

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The different parts of a problem are often unrelated.
- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.

Problem 1. A monkey types **8 digits** at random. (Each keystroke is independent of the others with all of the **10** possibilities 0,1,2,3,4,5,6,7,8,9 being equally likely.

(a) (13%) Let X_i denote the number of digits that the monkey types exactly i times. For example, if the monkey types 43266999 then $X_1 = 3$, $X_2 = 1$, $X_3 = 1$ and $X_i = 0$ for $i \geq 4$.

Find the probability distribution of X_4.

This problem is similar to problem 1.46 in the text. The monkey situation is equivalent to placing 8 balls at random into 10 cells. X4 can only take {2, 1, 0} 2

3 $X_4 = 2$ {4.4} $\binom{10}{2}\binom{8}{4}\binom{4}{4}$ 3150

$X_4 = 1$ {4.3.1} $\binom{10}{1}\binom{8}{4}\binom{9}{1}\binom{4}{3}\binom{8}{1}$ 201600

{4.2.2} $\binom{10}{1}\binom{8}{4}\binom{9}{2}\binom{4}{2}\binom{2}{2}$ 151200

5 {4.2.1.1} $\binom{10}{1}\binom{8}{4}\binom{9}{1}\binom{4}{2}\binom{8}{2}\binom{2}{1}$ 2116800.

{4.1.1.1.1} $\binom{10}{1}\binom{8}{4}\binom{9}{4}4!$ 2116800.

4586400.

3 $P(X_4 = 0) = 1 - P(X_4 = 2) - P(X_4 = 1)$

[Problem 1 continued]

(b) (7%) What is the probability the monkey types at least two 9's?

Parts (b) and (c) are similar to Problem 1.36. The monkey is performing 8 independent trials in which the probability of getting of success (typing a 9) is 1/10 on each trial.

P(at least two 9's) = 1 − P(types 0 9's) − P(types one 9's)

$$= 1 - \frac{9^8}{10^8} - \frac{\binom{8}{1} 9^7}{10^8}$$

$$= .1868$$

(c) (7%) What is the conditional probability the monkey types at least two 9's, given that it types at least one 9?

P(at least two 9's | at least one 9's)

$$= \frac{P(\text{at least two 9's})}{P(\text{at least one 9's})} = \frac{1 - \frac{9^8}{10^8} - \frac{\binom{8}{1}9^7}{10^8}}{1 - P(\text{type 0 9's})}$$

$$= \frac{1 - \frac{9^8}{10^8} - \frac{\binom{8}{1}9^7}{10^8}}{1 - \frac{9^8}{10^8}}$$

$$= .3282$$

(d) (3%) Consider the events:

$A = \{$the monkey types only odd digits$\}$ and $B = \{$the monkey types only even digits$\}$.

These are events are ... (Circle the correct completion.)

a) independent **b)** mutually exclusive **c)** both (a) and (b) **d)** neither (a) nor (b)

Part (d) is loosely inspired by Problem 1.39.

Problem 2. (14%) Suppose X has density (pdf) $f_X(x) = e^{-x}$ for $x > 0$.

Find the density (pdf) of $Y = e^X$. (Make sure to specify where it is positive.)

This is similar to the simple transformation problems in the parts of problems 2.1 and 2.2.

$Y = e^X \implies X = \log Y.$

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

$$= e^{-\log y} \cdot \left| \frac{1}{y} \right|$$

$$= \frac{1}{y} \cdot \frac{1}{y}$$

$$= \frac{1}{y^2} \qquad (y > 1)$$

$\left(\begin{array}{l} \because\ y = e^x.\quad x > 0. \\ \therefore\ y > 1. \end{array} \right)$

$\therefore$ The density of $Y = e^X$ is

$$f_Y(y) = \begin{cases} \frac{1}{y^2} & y > 1 \\ 0 & y \leq 1 \end{cases}$$

Problem 3. Suppose X has mass function (pmf) $f_X(x) = (e-1)e^{-x}$ for $x = 1, 2, 3, \ldots$

(a) (10%) Find the mass function (pmf) of $Y = e^X$.

This is somewhat similar to Problem 2.3.

The mass function of Y is

$$P(Y=y) = P(e^X = y) = P(X = \log y) = (e-1)e^{-\log y}$$

$$= \frac{(e-1)}{y}$$

where $y = e, e^2, e^3 \ldots$

(b) (3%) Let P_X denote the induced probability function of X.
What is the value of $\boldsymbol{P_X([1,2) \cup (4,5])}$?

This problem is only intended to see if you remember the definition of the "induced probability function" defined on page 12 of notes3.pdf, and used in various places later in notes3.pdf.

$P_X([1,2) \cup (4,5])$

$= P_X(X=1) + P_X(X=5)$

$= (e-1)e^{-1} + (e-1)e^{-5}$

In this problem we abbreviate events by omitting the intersection symbol $\cap$ so that $A \cap B = AB$, $A \cap B \cap C = ABC$, etc.

Problem 4. (10%) Let A, B, C, D be any events. (We are NOT assuming any special properties like independence.) Find an expression for $P(AB \cup AC \cup AD \cup BC)$ as a sum of probabilities of various intersections of these events. (Simplify your answer as much as possible.)

Use inclusion-exclusion. This is similar to the example on page 5 of notes3.pdf.

$$P(AB \cup AC \cup AD \cup BC)$$

$$= P(AB) + P(AC) + P(AD) + P(BC)$$

$$- P(AB \cap AC) - P(AB \cap AD) - P(AB \cap BC) - P(AC \cap AD) - P(AC \cap BC) - P(AD \cap BC)$$

$$+ P(AB \cap AC \cap AD) + P(AB \cap AD \cap BC) + P(AC \cap AD \cap BC) + P(AB \cap AC \cap BC)$$

$$- P(AB \cap AC \cap AD \cap BC)$$

$$= P(AB) + P(AC) + P(AD) + P(BC)$$

$$- P(ABC) - P(ABD) - P(ABC) - P(ACD) - P(ABC) - P(ABCD)$$

$$+ P(ABCD) + P(ABCD) + P(ABCD) + P(ABC)$$

$$- P(ABCD)$$

$$= P(AB) + P(AC) + P(AD) + P(BC)$$

$$- 2P(ABC) - P(ABD) - P(ACD)$$

$$+ P(ABCD)$$

Problem 5. Consider the "Gambler's Ruin" problem with a biased coin having probability p of Heads (win a dollar) and $1-p$ of Tails (lose a dollar). Let $\psi(z)$ denote the probability of reaching a given goal of g dollars starting with an initial fortune of z dollars.

(a) (10%) Use the Law of Total Probability to find an equation that $\psi(z)$ must satisfy. (Show the argument in detail.)

This is part of problem B7. You should give more detail than the official solution (page 19 of solutions1_ABexercises.pdf) which unfortunately only gives the answer. The more detailed argument is similar to that on page 18 of notes2.pdf.

$$\psi(z) = P(\underbrace{\text{reaching the goal with } z \text{ initial}}_{\text{Event } A})$$
$$= P(A, \underbrace{\text{first toss is head}}_{B_1}) + P(A, \underbrace{\text{first toss is tail}}_{B_2})$$
$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$$
$$= p\psi(z+1) + (1-p)\psi(z-1)$$

You are given that $\psi(z) = \dfrac{R^z - 1}{R^g - 1}$ where $R = (1-p)/p$. Using this, answer the following.

(b) (6%) What is the probability the gambler's first two tosses were heads, given that he reached the goal?

This is similar to the examples on page 21 of notes2.pdf.

$$P(\text{first two heads} \mid \text{reached goal})$$
$$= \frac{P(\text{reached goal} \mid \text{first two tosses were heads})\,P(\text{first two tosses were heads})}{P(\text{reached goal})}$$
$$= \frac{p^2\psi(z+2)}{\psi(z)}$$
$$= \frac{\frac{p^2(R^{z+2}-1)}{R^g-1}}{\frac{R^z-1}{R^g-1}} = \frac{p^2(R^{z+2}-1)}{R^z-1} = \frac{p^2\left(\frac{(1-p)^{z+2}}{p^{z+2}}-1\right)}{\frac{(1-p)^z}{p^z}-1} = \frac{(1-p)^{z+2}-p^z}{(1-p)^z-p^z}$$

Problem 6. Suppose $X_1, X_2, X_3, \ldots$ are independent Poisson(1) random variables. Define

$$Y = \sum_{i=1}^{\infty} \frac{X_i}{2^i} = \frac{X_1}{2} + \frac{X_2}{4} + \frac{X_3}{8} + \cdots.$$

Calculate the following.

This is similar to problem A3.

(a) (7%) $E(Y)$

$$E(Y) = E\left(\sum_{i=1}^{\infty} \frac{X_i}{2^i}\right) = E\left(\frac{X_1}{2}\right) + E\left(\frac{X_2}{4}\right) + E\left(\frac{X_3}{8}\right) + \cdots$$

$\because X_1, X_2 \cdots$ iid Poisson(1) $\quad \therefore E(X_i) = 1$

$$\therefore E(Y) = \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

(b) (6%) $\mathrm{Var}(Y)$

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\frac{X_i}{2^i}\right) = \sum_{i=1}^{\infty} \frac{1}{2^{2i}} \mathrm{Var}(X_i)$$

$\because \mathrm{Var}(X_i) = 1$

$$\therefore \mathrm{Var}(Y) = \sum_{i=1}^{\infty} \frac{1}{2^{2i}}$$

$$= \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$$

If you have finished everything else and have checked your answers, then try this:

Problem 7. (4%) Suppose X has density (pdf) $f_X(x) = e^{-x}$ for $x > 0$. Let $Y = \dfrac{\sin(X)}{X}$. Find $f_Y(0)$, the density of Y at zero.

This is most easily done using the formula on page 15 of notes4.pdf.