TEST #2
STA 5326
November 1, 2007

Name: Key

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the appendix "Table of Common Distributions" given in the back of the text. Use the appendix wherever possible.)
- Some problems may require you to compute an approximate answer, but do not explicitly state this. (You are expected to realize that an approximation is called for.)
- Show and explain your work (including your calculations) for all the problems unless you are specifically told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.

Problem 1. (15%) A lump of radioactive material contains 10^7 atoms of isotope A, 10^8 atoms of isotope B, and 10^9 atoms of isotope C. The probability that a particular individual atom decays during a one minute period is 1.3×10^{-7}, 4.0×10^{-9}, and 6.0×10^{-10} for isotopes A, B, and C, respectively. What is the probability that a total of exactly 3 atoms decay in the next minute?

dist^n of atom decays in A $\sim \text{Bin}(10^7, 1.3\times10^{-7})$

dist^n of atom decays in B $\sim \text{Bin}(10^8, 4.0\times10^{-9})$

dist^n of atom decays in C $\sim \text{Bin}(10^9, 6.0\times10^{-10})$

since all n_A, n_B, n_C all large and p_A, p_B, p_C small.

can use Poisson Approximation

$A \sim \text{Poisson}(10^7 \times 1.3\times10^{-7})$

$B \sim \text{Poisson}(10^8 \times 4.0\times10^{-9})$

$C \sim \text{Poisson}(10^9 \times 6.0\times10^{-10})$

$A+B+C \sim \text{Poisson}(n_Ap_A + n_Bp_B + n_Cp_C)$

$= \text{Poisson}(1.3+0.4+0.6=2.3)$

$P(A+B+C=3) = \dfrac{e^{-2.3}(2.3)^3}{3!} = 0.2033$

Problem 2. Al and Bob are each rolling a fair six-sided die. They decide to compete to see who will be the first to roll a thousand 1's. They start at the same time, and roll at exactly the same rate (one roll per second).

Let X be the number of rolls Al needs to get his thousand 1's.

(a) (5%) What is the distribution of X? (Give the name of the distribution and the values of any parameters. No work is required.)

$$X \sim \text{NegBin}(\underset{r}{\underset{\uparrow}{1000}}, \underset{p}{\underset{\uparrow}{\tfrac{1}{6}}})$$

(b) (4%) What are the mean and variance of X? (No work is required.)

2 $EX = \frac{r}{p} = \frac{1000}{1/6} = 6000$ $\leftarrow$ use NegBin dist$^{\underline{n}}$ in the Notes!

2 $Var(X) = \frac{r(1-p)}{p^2} = \frac{1000 \times \frac{5}{6}}{(1/6)^2} = 30{,}000.$

[Problem 2 continued]

(c) (10%) What is the probability that Bob finishes exactly one second before Al?

$X = $ # of rolls Al need to get 1000 1's

$Y = $ # of rolls Bob need to get 1000 1's

want to know $P(Y - X = 1)$

use normal approximation.

$X \sim N(6000, 30000)$ $\quad Y \sim N(6000, 3000)$

$D = Y - X \sim N(0, 60000)$

Let $D^* \sim N(0, 60000)$

$$P(D=1) \approx P(D^* \in (1-0.5, 1+0.5))$$
$$= P(0.5 < D^* < 1.5)$$
$$= P\left(\frac{0.5}{\sqrt{60000}} < Z < \frac{1.5}{\sqrt{60000}}\right)$$
$$\approx P(0.002 < Z < 0.006)$$

$= \Phi(0.006) - \Phi(0.002)$

by calculator

$= 0.0024 - 0.0008$

$= 0.0016$

$$\text{or} = \int_{0.002}^{0.006} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$
$$\approx (0.006 - 0.002) \cdot \frac{1}{\sqrt{2\pi}}$$
$$= 0.0016$$

Problem 3. (12%) Let $X \sim \text{Binomial}(\boldsymbol{n}, \boldsymbol{p})$ and $Y \sim \text{Beta}(\boldsymbol{\alpha = n-2, \beta = 3})$ where $n \geq 3$. Prove that

$$P(X \leq 2) = P(Y \leq 1-p)\,.$$

Refer to your appendix for the Binomial and Beta distributions.

[Hint: Use repeated integration by parts or differentiate both sides with respect to p. Recall that $\Gamma(k) = (k-1)!$ for positive integers k.]

$$P(X \leq 2) = \sum_{x=0}^{2} \binom{n}{x} p^x (1-p)^{n-x}$$

$$P(Y \leq 1-p) = \int_0^{1-p} \frac{\Gamma(n-2+3)}{\Gamma(n-2)\Gamma(3)} Y^{n-2-1}(1-Y)^{3-1}\,dY$$

$$= \frac{n!}{(n-3)!\,2!} \cdot \int_0^{1-p} Y^{n-3}(1-Y)^2\,dY$$

$$= \frac{n!}{(n-3)!\,2!} \int_0^{1-p} (1-Y)^2 \cdot \frac{1}{n-2}\,dY^{n-2}$$

$$= \frac{n!}{(n-3)!\,2!}\left[(1-Y)^2 \frac{1}{n-2} \cdot Y^{n-2}\Big|_0^{1-p} + \frac{2}{n-2}\int_0^{1-p}(1-Y)Y^{n-2}\,dY\right]$$

$$= \binom{n}{2}p^2(1-p)^{n-2} + \frac{n!}{(n-2)!}\int_0^{1-p}(1-Y)\frac{1}{n-1}\,dY^{n-1}$$

$$= \binom{n}{2}p^2(1-p)^{n-2} + \frac{n!}{(n-2)!}\left[(1-Y)\frac{1}{n-1}Y^{n-1}\Big|_0^{1-p} + \frac{1}{n-1}\int_0^{1-p}Y^{n-1}\,dY\right]$$

$$= \binom{n}{2}p^2(1-p)^{n-2} + \binom{n}{1}p(1-p)^{n-1} + \frac{n!}{(n-1)!}\int_0^{1-p}Y^{n-1}\,dY$$

$$= \binom{n}{2}p^2(1-p)^{n-2} + \binom{n}{1}p(1-p)^{n-1} + \frac{n!}{(n-1)!}\cdot\frac{1}{n}\,Y^n\Big|_0^{1-p}$$

$$= \binom{n}{2}p^2(1-p)^{n-2} + \binom{n}{1}p(1-p)^{n-1} + (1-p)^n$$

$$= \sum_{i=0}^{2}\binom{n}{i}p^i(1-p)^{n-i}$$

$$\therefore P(Y \leq 1-p) = P(X \leq 2) \qquad \#$$

Problem 4. (14%) Show that $E(X-a)^2$ is minimized when $a = EX$. (Assume that X has a finite mean and variance.)

Assume that X has density f(x).

$$E(X-a)^2 = \int_{-\infty}^{\infty} (x-a)^2 f(x)\,dx$$

$$\frac{d}{da} E(X-a)^2 = \frac{d}{da} \int_{-\infty}^{\infty} (x-a)^2 f(x)\,dx$$

$$= -\int_{-\infty}^{\infty} 2(x-a) f(x)\,dx$$

$$= 2a\int_{-\infty}^{\infty} f(x)\,dx - 2\int_{-\infty}^{\infty} x f(x)\,dx$$

$$= 2a - 2EX = 0$$

$$\Rightarrow EX = a$$

$$\frac{d^2}{da^2} E(X-a)^2 = \frac{d}{da}(2a-2EX) = 2 > 0.$$

So when $a = EX$. $E(X-a)^2$ achieves its minimum #

Problem 5.

(a) (7%) Write the integral that would define the moment generating function (mgf) $M(t)$ of the density (pdf) given by

$$f(x) = \begin{cases} \dfrac{2}{\pi(1+x^2)} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0\,. \end{cases}$$

$$\text{mgf} = \int_0^\infty e^{tx} \frac{2}{\pi(1+x^2)}\, dx$$

(b) (7%) For what values of t is this integral finite? (Justify your answer.)

when $t=0$, $M(t) = \int_0^\infty \frac{2}{\pi(1+x^2)}\, dx = 1$ finite.

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when $t<0$. $\frac{2e^{tx}}{\pi(1+x^2)} \to 0$ as $x \to \infty$ fast enough s.t. integral finite.

so $M(t)$ is finite.

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when $t>0$, $\frac{2e^{tx}}{\pi(1+x^2)} \to \infty$ as $x \to \infty$.

so integral $= \infty$.

In all. $M(t)$ finite for $t \leq 0$.

Problem 6. Suppose X_1, X_2, X_3, X_4 are iid Exponential(β) random variables and define $Y = \max(X_1, X_2, X_3, X_4)$.

(a) (7%) What is EY? (State the answer and give a brief justification.)

$$Y = X_{(1)} + [X_{(2)} - X_{(1)}] + [X_{(3)} - X_{(2)}] + [X_{(4)} - X_{(3)}]$$

$$\exp(\tfrac{\beta}{4}) \quad \exp(\tfrac{\beta}{3}) \quad \exp(\tfrac{\beta}{2}) \quad \exp(\beta) \quad \text{independent}$$

$$\text{so } EY = \frac{\beta}{4} + \frac{\beta}{3} + \frac{\beta}{2} + \beta$$

$$= \frac{25}{12}\beta.$$

or other way:

(a) · pdf $f_{max}(t) = 4(F(t))^3 f(t) = 4(1-e^{-t/\beta})^3 \frac{1}{\beta} e^{-t/\beta}$.

$$EY = \int_0^\infty t \cdot 4(1-e^{-t/\beta})^3 \frac{1}{\beta} e^{-t/\beta}\, dt$$

$$= \frac{4}{\beta}\int_0^\infty t(1-e^{-3t/\beta} - 3e^{-t/\beta} + 3e^{-2t/\beta})e^{-t/\beta}\,dt$$

$$= \frac{4}{\beta}\int_0^\infty (te^{-t/\beta} - te^{-4t/\beta} - 3te^{-2t/\beta} + 3te^{-3t/\beta})\,dt \rightarrow \text{Back}$$

(b) (7%) What is Var(Y)? (State the answer and give a brief justification.)

$$Var(Y) = Var(X_{(1)}) + Var(X_{(2)} - X_{(1)}) + Var(X_{(3)} - X_{(2)}) + Var(X_{(4)} - X_{(3)})$$

$$= \left(\frac{\beta}{4}\right)^2 + \left(\frac{\beta}{3}\right)^2 + \left(\frac{\beta}{2}\right)^2 + (\beta)^2$$

$$= \frac{205}{144}\beta^2$$

For each item.

$$\int_0^\infty t e^{-kt/\beta} dt = \int_0^\infty t \cdot (-\tfrac{\beta}{k}) d e^{-kt/\beta}$$
$$= t(-\tfrac{\beta}{k}) e^{-kt/\beta} \Big|_0^\infty + \tfrac{\beta}{k} \int_0^\infty e^{-kt/\beta} dt$$
$$= -\tfrac{\beta^2}{k^2} e^{-kt/\beta} \Big|_0^\infty$$
$$= \tfrac{\beta^2}{k^2}$$

so $E(Y) = \frac{4}{\beta} \left(\beta^2 - \frac{\beta^2}{4^2} - \frac{3\beta^2}{2^2} + \frac{3\beta^2}{3^2}\right) = \frac{25}{12}\beta.$

(b) same way

$E(Y^2)$. we have to calculate.

$$\int_0^\infty t^2 e^{-kt/\beta} dt = \int_0^\infty t^2 \cdot (-\tfrac{\beta}{k}) d e^{-\frac{kt}{\beta}}$$
$$= t^2 (-\tfrac{\beta}{k}) e^{-\frac{kt}{\beta}} \Big|_0^\infty + \tfrac{\beta}{k} \int_0^\infty 2t\, e^{-\frac{kt}{\beta}} dt$$
$$= \tfrac{\beta}{k} \cdot 2 \cdot \tfrac{\beta^2}{k^2}$$
$$= 2\beta^3/k^3$$

so $E(Y^2) = \frac{4}{\beta}\left(2\beta^3 - \frac{2\beta^3}{4^3} - \frac{2\times 3\beta^3}{2^3} + \frac{2\times 3\beta^3}{3^3}\right) = \frac{415}{72}$

$$\mathrm{var}(Y) = E(Y^2) - (E(Y))^2$$
$$= \left(\tfrac{830}{144} - \tfrac{625}{144}\right)\beta^2$$
$$= \tfrac{205}{144}\beta^2.$$

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Problem 7. (12%) Suppose $X \sim \text{Poisson}(\lambda = n)$. What is the moment generating function (mgf) of $Y = \dfrac{X-n}{\sqrt{n}}$. (Use the appendix for the Poisson mgf.)

$$M_X(t) = e^{\lambda(e^t-1)} = e^{n(e^t-1)}$$

$$M_Y(t) = M_{\frac{X-n}{\sqrt{n}}}(t) = M_{\frac{X}{\sqrt{n}}-\sqrt{n}}(t)$$

$$= e^{-\sqrt{n}t} M_{\frac{X}{\sqrt{n}}}(t)$$

$$= e^{-\sqrt{n}t} \cdot M_X\left(\frac{t}{\sqrt{n}}\right)$$

$$= e^{-\sqrt{n}t} \cdot e^{n(e^{\frac{t}{\sqrt{n}}}-1)}$$

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