TEST #2
STA 5326 **Name:**____________________________________
November 1, 2007

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the appendix "Table of Common Distributions" given in the back of the text. Use the appendix wherever possible.)
- Some problems may require you to compute an approximate answer, but do not explicitly state this. (You are expected to realize that an approximation is called for.)
- Show and explain your work (including your calculations) for all the problems unless you are specifically told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.

Problem 1. (15%) A lump of radioactive material contains 10^7 atoms of isotope A, 10^8 atoms of isotope B, and 10^9 atoms of isotope C. The probability that a particular individual atom decays during a one minute period is 1.3×10^{-7}, 4.0×10^{-9}, and 6.0×10^{-10} for isotopes A, B, and C, respectively. What is the probability that a total of exactly 3 atoms decay in the next minute?

This is similar to exercise C2. Use a Poisson approximation as on page 30 of notes7.pdf.

Problem 2. Al and Bob are each rolling a fair six-sided die. They decide to compete to see who will be the first to roll a thousand 1's. They start at the same time, and roll at exactly the same rate (one roll per second).

Let X be the number of rolls Al needs to get his thousand 1's.

(a) (5%) What is the distribution of X? (Give the name of the distribution and the values of any parameters. No work is required.)

You are waiting for 1000 successes (getting a 1) in a sequence of Bernoulli trials (rolls of the dice). The probability of success on each trial is 1/6. Thus $X \sim$ *Neg.Bin*$(r = 1000, p = 1/6)$. *Note that this is the "lecture" version of the Negative Binomial distribution, not the "textbook" version.*

(b) (4%) What are the mean and variance of X? (No work is required.)

Note: the mean of the "lecture" version of the Negative Binomial is r/p.

[Problem 2 continued]

(c) (10%) What is the probability that Bob finishes exactly one second before Al?

This is similar to exercise C5 (the voting problem), but using the Negative Binomial instead of the Binomial distribution. See the solution to C5 or (better yet) the lecture notes on pages 20 to 24 of notes8.pdf. Of course, we must take μ *and* σ^2 *to be the values for the Negative Binomial instead of the Binomial distribution. Recall (see page 19 of notes8.pdf) that the Negative Binomial*(r, p) *distribution is approximately normal when* r *is sufficiently large.*

Problem 3. (12%) Let $X \sim \text{Binomial}(\boldsymbol{n}, \boldsymbol{p})$ and $Y \sim \text{Beta}(\boldsymbol{\alpha = n - 2}, \boldsymbol{\beta = 3})$ where $n \geq 3$. Prove that

$$P(X \leq 2) = P(Y \leq 1 - p) .$$

Refer to your appendix for the Binomial and Beta distributions.

[Hint: Use repeated integration by parts or differentiate both sides with respect to p. Recall that $\Gamma(k) = (k-1)!$ for positive integers k.]

This problem is a special case of Exercise 2.40. Integration by parts is probably the most straightforward approach, but the differentiation approach (as in the old student solution) also works.

Problem 4. (14%) Show that $E(X-a)^2$ is minimized when $a = EX$.
(Assume that X has a finite mean and variance.)

This is problem 2.19. Two different solutions are given in the chapter 2 solutions posted in mordor.

Problem 5.

This is similar to exercise 3.21, except we are using a "half Cauchy" distribution. See page 5 of notes6.pdf for some relevant discussion. Unfortunately, the solution given in the solution manual is not very good and has a typo: $e^{tx} > tx$ *on* $(0, \infty)$.

(a) (7%) Write the integral that would define the moment generating function (mgf) $M(t)$ of the density (pdf) given by

$$f(x) = \begin{cases} \dfrac{2}{\pi(1+x^2)} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0\,. \end{cases}$$

(b) (7%) For what values of t is this integral finite? (Justify your answer.)

Problem 6. Suppose X_1, X_2, X_3, X_4 are iid Exponential(β) random variables and define $Y = \max(X_1, X_2, X_3, X_4)$.

This is similar to the lightbulb example (the "k stall" problem) in lecture. See page 12 of notes8.pdf

(a) (7%) What is EY? (State the answer and give a brief justification.)

(b) (7%) What is $\mathrm{Var}(Y)$? (State the answer and give a brief justification.)

Problem 7. (12%) Suppose $X \sim \text{Poisson}(\lambda = n)$. What is the moment generating function (mgf) of $Y = \dfrac{X - n}{\sqrt{n}}$. (Use the appendix for the Poisson mgf.)

This is taken from pages 22 and 23 of notes6.pdf and uses the scaling properties on page 10 of notes6.pdf (or can be done directly from the definition of mgf).