TEST #3
STA 5326
December 10, 2007

Name:

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the appendix "Table of Common Distributions" given in the back of the text. Use the appendix wherever possible.)
- **If you are asked to find a joint density, you should always specify the support. That is, if your anwswer only applies in a certain region (and the density is zero otherwise), this region should be clearly specified.**
- Show and explain your work (including your calculations) for all the problems unless you are specifically told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** pages.

Problem 1. Let X and Y be independent random variables with densities

$$f_X(x) = \frac{e^{-x^2}}{\sqrt{\pi}}, \quad -\infty < x < \infty \quad \text{and} \quad f_Y(y) = e^{-y}, \; 0 \le y < \infty$$

and define

$$U = X^2 + Y \quad \text{and} \quad W = \frac{Y}{\sqrt{X^2+Y}}.$$

(a) (14%) Find the joint density of U and W.

X, Y joint density $f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{e^{-x^2}}{\sqrt{\pi}} \cdot e^{-y} = \frac{1}{\sqrt{\pi}} e^{-(x^2+y)}$

support: $\{-\infty < x < \infty, \; 0 \le y < \infty\}$

Since $U = x^2 + Y$, $W = \frac{Y}{\sqrt{x^2+Y}}$ $\Rightarrow Y = W\sqrt{u}$ $\quad X^2 = U - W\sqrt{U}$

$(X, Y) \to (U, W)$ is not one-to-one map, divide (X, Y) into three area.

$A_1 = \{x = 0, \; 0 \le y < \infty\}$, $\quad A_2 = \{X > 0, \; 0 \le y < \infty\}$ $\quad A_3 = \{X < 0, \; 0 \le y < \infty\}$

$\Downarrow$ $\quad Y = w\sqrt{u}, \; X = \sqrt{U - w\sqrt{u}}$ $\qquad \Downarrow$ $\quad Y = w\sqrt{u}, \; x = -\sqrt{u - w\sqrt{u}}$

$$J_1 = \begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial Y}{\partial U} \\ \frac{\partial X}{\partial W} & \frac{\partial Y}{\partial W} \end{vmatrix} = \begin{vmatrix} \frac{1 - \frac{w}{2\sqrt{u}}}{2\sqrt{u - w\sqrt{u}}} & \frac{w}{2\sqrt{u}} \\ \frac{-\sqrt{u}}{2\sqrt{u - w\sqrt{u}}} & \sqrt{u} \end{vmatrix} = \frac{\sqrt{u}}{2\sqrt{u - w\sqrt{u}}} \qquad (x > 0)$$

$$J_2 = \frac{-\sqrt{u}}{2\sqrt{u - w\sqrt{u}}} \qquad (x < 0)$$

Joint density of (U, W)

$$f_{U,W} = \frac{1}{\sqrt{\pi}} e^{-u} \frac{\sqrt{u}}{\sqrt{u - w\sqrt{u}}}, \quad \text{support: } \{0 \le U < \infty, \; 0 \le W < \infty\}$$

also, $u - w\sqrt{u} > 0 \Rightarrow W < \sqrt{u}$

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(b) (4%) Are U and W independent? (Justify your answer.)

U and W are not independent, because U and W can't be factored as a product of two seperate distribution.

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Problem 2. Let X_1, X_2, X_3 be iid with pdf

$$f(x) = \frac{x}{2} \quad \text{if } 0 < x < 2 \quad (\text{and } f(x) = 0 \text{ otherwise})$$

and let Y_1, Y_2, Y_3 denote the order statistics: $Y_1 = X_{(1)}$, $Y_2 = X_{(2)}$, $Y_3 = X_{(3)}$.

(a) (5%) What is the joint density of (Y_1, Y_2, Y_3)? (No work is required. Just state the answer.)

$$f_{Y_1,Y_2,Y_3} = 3!\, f(y_1) f(y_2) f(y_3) \mathbb{1}_{(y_1<y_2<y_3)}$$
$$= 3!\, \frac{y_1}{2} \cdot \frac{y_2}{2} \cdot \frac{y_3}{2} \mathbb{1}_{(y_1<y_2<y_3)} = \frac{3}{4} y_1 y_2 y_3 \mathbb{1}_{(y_1<y_2<y_3)}$$

support of f_{Y_1,Y_2,Y_3} is $\{0<y_1<y_2<y_3<2\}$

(b) (12%) Let $Z_1 = Y_1/Y_2$, $Z_2 = Y_2/Y_3$, and $Z_3 = Y_3$. Find the joint density of (Z_1, Z_2, Z_3).

$Y_3 = Z_3$, $Y_2 = Z_2 Z_3$, $Y_1 = Z_1 Z_2 Z_3$

This is one-to-one map.

$$J = \begin{vmatrix} \frac{\partial Y_1}{\partial Z_1} & \frac{\partial Y_2}{\partial Z_1} & \frac{\partial Y_3}{\partial Z_1} \\ \frac{\partial Y_1}{\partial Z_2} & \frac{\partial Y_2}{\partial Z_2} & \frac{\partial Y_3}{\partial Z_2} \\ \frac{\partial Y_1}{\partial Z_3} & \frac{\partial Y_2}{\partial Z_3} & \frac{\partial Y_3}{\partial Z_3} \end{vmatrix} = \begin{vmatrix} Z_2 Z_3 & 0 & 0 \\ Z_1 Z_3 & Z_3 & 0 \\ Z_1 Z_2 & Z_2 & 1 \end{vmatrix} = Z_2 Z_3^2$$

∴ Joint density of Z_1, Z_2, Z_3 is $3!\left(\frac{Z_1 Z_2 Z_3}{2}\right)\left(\frac{Z_2 Z_3}{2}\right)\left(\frac{Z_3}{2}\right)|J|\, \mathbb{1}_{(Z_1 Z_2 Z_3 < Z_2 Z_3 < Z_3)}$

$$= \frac{3}{4} z_1 z_2^3 z_3^5 \mathbb{1}_{(z_1 z_2 z_3 < z_2 z_3 < z_3)}$$

Support of f_{Z_1,Z_2,Z_3} is $\{0<Z_1<1,\ 0<Z_2<1,\ 0<Z_3<2\}$

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Problem 3. The random pair (X, Y) has a joint pmf given in the following table.

		X = 1	2	3
	1	0	1/4	1/4
Y	2	1/6	0	1/6
	3	1/12	1/12	0

(a) (6%) Find the marginal mass function (pmf) of X

$p(X=1) = p(X=1, Y=1) + p(X=1, Y=2) + p(X=1, Y=3) = 0 + \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$

$p(X=2) = p(X=2, Y=1) + p(X=2, Y=2) + p(X=2, Y=3) = \frac{1}{4} + 0 + \frac{1}{12} = \frac{1}{3}.$

$p(X=3) = p(X=3, Y=1) + p(X=3, Y=2) + p(X=3, Y=3) = \frac{1}{4} + \frac{1}{6} + 0 = \frac{5}{12}$

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(b) (6%) Compute $E(Y^3 \mid X = 2)$.

$P_{Y|X} = \frac{P_{X,Y}}{P_X} \Rightarrow p(Y=1 \mid X=2) = \frac{p(X=2, Y=1)}{p(X=2)} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$

$p(Y=2 \mid X=2) = \frac{p(X=2, Y=2)}{p(X=2)} = \frac{0}{\frac{1}{3}} = 0$

$p(Y=3 \mid X=2) = \frac{p(X=2, Y=3)}{p(X=2)} = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$

$\therefore E(Y^3 \mid X=2) = 1^3 \cdot p(Y=1 \mid X=2) + 2^3 \cdot p(Y=2 \mid X=2) + 3^3 \, p(Y=3 \mid X=2)$

$= 1 \cdot \frac{3}{4} + 8 \cdot 0 + 27 \cdot \frac{1}{4} = \frac{30}{4} = \frac{15}{2}$

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Problem 4.

(a) (4%) State the definition of a one parameter exponential family.

$$h(x) \cdot c(\theta) \exp\{a(\theta) \cdot t(x)\}$$

(b) (8%) Show that the family of Binomial(n, p) distributions with n known (fixed) is a one parameter exponential family.

fixed

$$P(X=x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots$$

$$= \binom{n}{x}(1-p)^n \left(\frac{p}{1-p}\right)^x$$

$$= \binom{n}{x}(1-p)^n \cdot \exp\left[x \cdot \log\left(\frac{p}{1-p}\right)\right]$$

$$= \underbrace{\binom{n}{x} \mathbb{1}_{\{x=0,1,2,\ldots\}}}_{h(x)} \underbrace{(1-p)^n}_{c(\theta)} \exp\{\underbrace{x}_{t(x)} \cdot \underbrace{\log\left(\frac{p}{1-p}\right)}_{a(\theta)}\}.$$

So we know Binomial (n, p) is one parameter exponential family.

(c) (4%) If neither n or p is known (they are both allowed to vary), does the family of Binomial(n, p) distributions form an exponential family? (Justify your answer.)

NO, If n not fixed, $\{x = 0, \ldots n\}$

so we can know $h(x)$ can not expressed as a function of x alone.

Problem 5. The function $f(x) = \dfrac{140x^3}{(1+x)^8}$ for $x > 0$ (and $f(x) = 0$ otherwise) is a pdf (density) which satisfies

$$\int_{\mathbf{2.3}}^{\infty} f(x)\,dx = \mathbf{.13} \quad \text{and} \quad \int_{-\infty}^{\infty} x f(x)\,dx = \frac{4}{3}.$$

Suppose that X is a random variable with density $\frac{1}{7} f\left(\frac{x-5}{7}\right)$.

(a) (10%)Find the value c such that $P(X > c) = \mathbf{.13}$.

Since X have a location-Scale density.

and $\int_{2.3}^{\infty} f(x)dx = 1.3$.

$P(X>c) = \int_c^{\infty} \frac{1}{7} f(\frac{x-5}{7}) dx$ Let $\frac{x-5}{7} = z$

$= \int_{\frac{c-5}{7}}^{\infty} \frac{1}{7} f(z) \cdot 7 \cdot dz$

$= \int_{\frac{c-5}{7}}^{\infty} f(z) dz = 0.13$. $\Rightarrow \frac{c-5}{7} = 2.3 \Rightarrow c = 21.1$

(b) (5%) Find EX.

$EX = \int_{-\infty}^{\infty} x \cdot \frac{1}{7} f(\frac{x-5}{7}) dx$ Let $\frac{x-5}{7} = z$

$= \int_{-\infty}^{\infty} (7z+5) \cdot \frac{1}{7} f(z) \cdot 7 \cdot dz$

$= \int_{-\infty}^{\infty} (7z) f(z) dz + \int_{-\infty}^{\infty} 5 \cdot f(z) dz$

$= \frac{4}{3} \cdot 7 + 5 = \frac{43}{3}$.

Problem 6. A packet of flower seeds contains $R+B$ seeds with R seeds of a red flowering variety and B seeds of a blue flowering variety. If a seed is planted (and properly cared for), it will produce flowers with probability p. Assume the seeds grow independently of each other. Suppose that k seeds are selected at random from the packet and planted. Let X be the number of plants which produce red flowers.

> **Hint:** Introduce the random variable $Y =$ the number of "red seeds" planted, and consider the conditional distribution of X given Y.

(a) (6%) Find EX.

Define Y = the number of "red seeds" planted, $P(Y=y) = \underbrace{\dfrac{\binom{R}{y}\binom{B}{k-y}}{\binom{R+B}{k}}}_{\text{Hypergeometric distribution}}$

$X|Y = \text{Binomial}(Y, p)$

$EY = \dfrac{RK}{(R+B)}$, $E(X|Y) = Yp$.

$EX = E[E(X|Y)] = E[Yp] = \dfrac{Rkp}{(R+B)}$.

(b) (6%) Find Var(X).

$\text{Var}(X) = \text{Var}[E[X|Y]] + E[\text{Var}(X|Y)]$

$= \text{Var}[Yp] + E[Yp(1-p)]$

$= p^2 \cdot \text{Var}(Y) + p(1-p) \cdot E(Y)$

$= p^2 \cdot \dfrac{KR}{(R+B)} \cdot \dfrac{B \cdot (R+B-k)}{(R+B)(R+B-1)} + p(1-p) \cdot \dfrac{RK}{(R+B)}$.

In the following two problems, very little work is required. State the correct answer, and show a **little** work.

Problem 7. (5%) Suppose the random variables X and Y have cdf's and joint cdf given by

$$F_X(x) = x^2, \quad F_Y(y) = y, \qquad \text{and} \quad F_{X,Y}(x,y) = x^2 y$$

for $0 \le x \le 1$ and $0 \le y \le 1$.

What is the value of $P(.3 \le X \le .7, .2 \le Y \le .9)$?

$P(0.3 \le X \le 0.7, 0.2 \le Y \le 0.9)$

$= F_{X,Y}(0.7, 0.9) - F_{X,Y}(0.7, 0.2) - F_{X,Y}(0.3, 0.9) + F_{X,Y}(0.3, 0.2)$

$= (0.7)^2 \cdot 0.9 - (0.7)^2 \cdot 0.2 - (0.3)^2 \cdot 0.9 + (0.3)^2 \cdot 0.2$

$= (0.7)^3 - (0.3)^2 \cdot 0.7$

$= 0.7 [(0.7)^2 - (0.3)^2]$

$= 0.7 \cdot 0.4 = 0.28$

Problem 8. (5%) Suppose $X \sim \text{Poisson}(\lambda = 5)$ and $Y = X + 3$. What is the value of $\text{Cov}(X, Y)$?

$\text{Cov}(X,Y) = \text{Cov}(X, X+3) = \text{Cov}(X,X) = \text{Var}(X) = \lambda = 5$