TEST #3
STA 5326 **Name:**__________________________________
December 10, 2007

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the appendix "Table of Common Distributions" given in the back of the text. Use the appendix wherever possible.)
- **If you are asked to find a joint density, you should always specify the support. That is, if your anwswer only applies in a certain region (and the density is zero otherwise), this region should be clearly specified.**
- Show and explain your work (including your calculations) for all the problems unless you are specifically told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** pages.

Problem 1. Let X and Y be independent random variables with densities

$$f_X(x) = \frac{e^{-x^2}}{\sqrt{\pi}}\,, \quad -\infty < x < \infty \quad \text{and} \quad f_Y(y) = e^{-y}\,, \quad 0 \le y < \infty$$

and define

$$U = X^2 + Y \quad \text{and} \quad W = \frac{Y}{\sqrt{X^2+Y}}\,.$$

(a) (14%) Find the joint density of U and W.

This problem sort of combines 4.20 and 4.24. The transformation is not one-to-one since the points (x, y) and $(-x, y)$ map to the same point (u, w). See page 1 of `mult_trans_order_stat.pdf`. *We must divide the plane into the left and right half-planes; on each half-plane, the mapping is smooth and one-to-one. The joint density is a sum of two terms (one for each half-plane) which turn out to be identical, so that the net effect of the many-to-oneness of the transformation is to put a factor of 2 in the answer.*

(b) (4%) Are U and W independent? (Justify your answer.)

Problem 2. Let X_1, X_2, X_3 be iid with pdf

$$f(x) = \frac{x}{2} \quad \text{if } 0 < x < 2 \quad (\text{and } f(x) = 0 \text{ otherwise})$$

and let Y_1, Y_2, Y_3 denote the order statistics: $Y_1 = X_{(1)}$, $Y_2 = X_{(2)}$, $Y_3 = X_{(3)}$.

This problem is a special case of 5.25.

(a) (5%) What is the joint density of (Y_1, Y_2, Y_3)? (No work is required. Just state the answer.)

The joint density is a special case of the formula near the bottom of page 230 of the text.

(b) (12%) Let $Z_1 = Y_1/Y_2$, $Z_2 = Y_2/Y_3$, and $Z_3 = Y_3$. Find the joint density of (Z_1, Z_2, Z_3).

Problem 3. The random pair (X, Y) has a joint pmf given in the following table.

		X		
		1	2	3
	1	0	1/4	1/4
Y	2	1/6	0	1/6
	3	1/12	1/12	0

This problem is similar to the discrete table example used in lecture.

(a) (6%) Find the marginal mass function (pmf) of X

(b) (6%) Compute $E\left(Y^{\mathbf{3}} \mid X = 2\right)$.

See page 11 of notes10.pdf.

Problem 4.

See page 6 of notes9.pdf.

(a) (4%) State the definition of a one parameter exponential family.

(b) (8%) Show that the family of Binomial(n, p) distributions with n known (fixed) is a one parameter exponential family.

(c) (4%) If neither n or p is known (they are both allowed to vary), does the family of Binomial(n, p) distributions form an exponential family? (Justify your answer.)

See page 10 of notes9.pdf. Every member of an exponential family of distributions must have the same support. Since the support of the Binomial(n, p) distribution is $\{0, 1, \ldots, n\}$ which depends on n, this cannot be an exponential family.

Problem 5. The function $f(x) = \dfrac{140x^3}{(1+x)^8}$ for $x > 0$ (and $f(x) = 0$ otherwise) is a pdf (density) which satisfies

$$\int_{\mathbf{2.3}}^{\infty} f(x)\,dx = \mathbf{.13} \quad \text{and} \quad \int_{-\infty}^{\infty} x f(x)\,dx = \frac{4}{3}\,.$$

Suppose that X is a random variable with density $\dfrac{1}{7} f\left(\dfrac{x-5}{7}\right)$.

(a) (10%)Find the value c such that $P(X > c) = \mathbf{.13}$.

This is a special case of exercise 3.38.

(b) (5%) Find EX.

We know from lecture (see page 1 of notes9.pdf) that $X \stackrel{d}{=} 5+7Z$ *where* Z *has pdf* f *(the "standard" pdf) so that* $EX = 5 + 7EZ = 5 + 7(4/3)$. *One can also calculate* EX *by writing the expected value as an integral, and then making the substitution* $z = (x-5)/7$.

Problem 6. A packet of flower seeds contains $R + B$ seeds with R seeds of a red flowering variety and B seeds of a blue flowering variety. If a seed is planted (and properly cared for), it will produce flowers with probability p. Assume the seeds grow independently of each other.
Suppose that k seeds are selected at random from the packet and planted. Let X be the number of plants which produce red flowers.

> **Hint:** Introduce the random variable Y = the number of "red seeds" planted, and consider the conditional distribution of X given Y.

See pages 16 and 17 of notes12.pdf.

(a) (6%) Find EX.

(b) (6%) Find $\mathrm{Var}(X)$.

In the following two problems, very little work is required. State the correct answer, and show a **little** work.

Problem 7. (5%) Suppose the random variables X and Y have cdf's and joint cdf given by

$$F_X(x) = x^2\,, \quad F_Y(y) = y\,, \qquad \text{and} \quad F_{X,Y}(x,y) = x^2 y$$

for $0 \le x \le 1$ and $0 \le y \le 1$.

What is the value of $P(.3 \le X \le .7\,,\ .2 \le Y \le .9)$?

Just use the result of exercise 4.9. (I am not asking you to prove this result since that would involve more than a "little" work.)

$$\begin{aligned} P(.3 \le X \le .7\,,\ .2 \le Y \le .9) &= P(.3 \le X \le .7)\, P(.2 \le Y \le .9) \\ &= ((.7)^2 - (.3)^2)\,(.9 - .2) \end{aligned}$$

Problem 8. (5%) Suppose $X \sim \text{Poisson}(\lambda = 5)$ and $Y = X + 3$. What is the value of $\text{Cov}(X, Y)$?