TEST #1
STA 5326 **Name:**___________________________________
September 25, 2008

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all the problems except Problem 6. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.
- There are a total of **104** points.

Problem 1. A dice game is played as follows. A bucket of dice contains 7 red dice, 7 green dice, and 7 blue dice, for a total of 21 dice. All of the dice are fair and 6-sided. The bucket is well mixed and a player, with his eyes closed, reaches in and grabs **4** dice. All 4 dice are simultaneously thrown. A throw of the dice receives points if (*i*) any numbers are repeated, (*ii*) all 4 dice are the same color, or (*iii*) the numbers can be arranged in a consecutive sequence (for example: 2,3,4,5). For a single throw of the dice, compute the following.

This is (somewhat) similar to exercise B5.

(a) (7%) What is the probability there are **no** repeated numbers?

It is intuitively clear that the colors of the dice are independent of the numbers showing on the dice. (This is an implicit assumption of the problem.) So we can essentially ignore most of the story here, and just compute the probability of no repeated numbers when rolling 4 dice (or rolling a single die 4 times, it makes no difference). The probability is

$$\frac{6 \cdot 5 \cdot 4 \cdot 3}{6^4} = \frac{\binom{6}{4} 4!}{6^4}$$

since the first die can be any of the 6 numbers, the second die any of the 5 numbers not equal to the first die, the third die any of the 4 numbers different from the first two dice, etc. The right hand form of the answer comes from a different counting argument: choose 4 distinct numbers to appear on the dice in $\binom{6}{4}$ *ways, and then assign them to the 4 dice in* $4!$ *ways.*

(b) (7%) What is the probability all 4 dice are the same color?

Here we can ignore the numbers showing on the dice. The probability of drawing 4 dice of the same color is

$$\frac{3 \cdot \binom{7}{4}}{\binom{21}{4}} = \frac{21 \cdot 6 \cdot 5 \cdot 4}{21 \cdot 20 \cdot 19 \cdot 18}.$$

The left hand form comes from choosing the color in 3 ways, and then choosing 4 of the 7 dice of that color. Here we are adopting an unordered viewpoint, regarding the 4 dice as an unordered sample chosen from the 21 dice. In the right hand answer, we regard the dice as being chosen in order; the first die, then the second, etc. The first die can be any of the 21, then the second must be one of the remaining 6 dice having the same color as the first, etc.

[**Problem 1 continued**]

(c) (7%) What is the probability the numbers can be arranged in a consecutive sequence?

The answer is

$$\frac{3 \cdot 4!}{6^4}$$

because there are 3 possible sets of consecutive numbers $\{1,2,3,4\}$, $\{2,3,4,5\}$, *and* $\{3,4,5,6\}$, *and each of these sets can be ordered in* $4!$ *different ways.*

(d) (7%) What is the probability the throw is completely worthless (receives zero points)?

The students may use without proof the result of exercise B4, but they must give a clear statement of this result.

{*Worthless*} $= A \cap B^c \cap C^c$ *where* A *is "no repeated numbers",* B *is "all same color", and* C *is "numbers form a consecutive sequence". Since the colors are independent of the numbers, we may simplify this as* $P(A \cap B^c \cap C^c) = P(A \cap C^c)P(B^c)$. *(This can***not** *be done in exercise B5.)* $P(B^c) = 1 - P(B)$ *is given in part (b). Then note that*

$$P(A \cap C^c) = P(A) - P(A \cap C) = P(A) - P(C)$$

since $C \subset A$ *(i.e.,* C *implies* A*) so that* $A \cap C = C$. *Thus the answer is*

$$P(\mathit{worthless}) = [P(A) - P(C)][1 - P(B)]$$

where $P(A)$, $P(B)$, $P(C)$ *are the answers to parts (a), (b), (c).*
This answer, or anything equivalent, should be given full credit (if accompanied by sufficient explanation) even if the answers to the previous parts are all wrong.

Problem 2. (15%) Find the pdf of Y when $Y = e^{-X}$ and X has the pdf

$$f_X(x) = 30x^2(1-x)^2\,, \quad 0 < x < 1.$$

This is similar to exercise 2.2 and other Section 2.1 exercises.

Problem 3. (15%) Find the pdf of Y when X has pdf $f_X(x) = e^{-x}$ for $x > 0$, and

$$Y = \begin{cases} 2X & \text{for } X \le 4 \\ 12 - X & \text{for } 4 < X \le 7 \\ 3X - 16 & \text{for } X > 7\,. \end{cases}$$

This is similar to the example on pages 27–29 of notes4.pdf.

Problem 4. There are 80 students at a fraternity party, with equal numbers of freshmen, sophomores, juniors, and seniors (20 of each). Eight people are randomly selected to receive door prizes.

This is similar to exercise 1.22.

(a) (8%) What is the probability these eight prizes are equally distributed to the four classes (two to each)?

(b) (8%) What is the probability that **no** freshmen receive prizes?

Problem 5. In a population, the proportion of good and bad drivers is $2/3$ and $1/3$, respectively. In any given year, a good driver will have an accident with probability $1/4$, and a bad driver with probability $3/4$. A driver is selected at random from this population and observed for 3 years.

This is similar to the "tossing a randomly selected coin" examples used in lecture. See pages 13–16 of notes2.pdf and pages 6–8 of notes3.pdf.

(a) (8%) What is the probability the driver has accidents in exactly 2 of these 3 years?

(b) (8%) If the driver has accidents in exactly 2 of the 3 years, what is the probability the driver is bad?

Problem 6. (10%) The random variable X has cdf

$$F(x) = \begin{cases} 0 & \text{for } x \;?\; 0 \\ x + 0.10 & \text{for } 0 \;?\; x \;?\; 0.3 \\ x + 0.25 & \text{for } 0.3 \;?\; x \;?\; 0.6 \\ 1 & \text{for } 0.6 \;?\; x \end{cases}$$

where each question mark (?) stands for either $<$ or $\leq$ (but I'm not telling you which).
Find values for the following. (No work is required. Just fill in the blanks. They are worth 2% each.)

$\boldsymbol{F(0.3)} =$ ____________________

$\boldsymbol{P(0 < X < 0.6)} =$ ____________________

$\boldsymbol{P(X = 0.3)} =$ ____________________

$\boldsymbol{P(X \geq 1)} =$ ____________________

$\boldsymbol{P(X = 0.5)} =$ ____________________

If you have finished everything else and have checked your answers, then try this. Partial credit will be minimal. Your solution must be nearly correct to receive any credit.

Problem 7. (4%) A three-legged alien has a closet containing n triplets of shoes. If $3k$ shoes are chosen at random ($k < n$), what is the probability that there will be no matching triplet in the sample?

Let A_i be the event that the sample contains all three members of triplet i. Then the probability of no matching triples is $1 - P(A_1 \cup A_2 \cup \cdots \cup A_n)$ which may be obtained by an inclusion-exclusion argument if we note that (for $j \leq k$)

$$P(A_1 \cap A_2 \cap \cdots \cap A_j) = \frac{\binom{3n-3j}{3k-3j}}{\binom{3n}{3k}}.$$

This leads to

$$P(\textit{no matching triple}) = 1 - \sum_{j=1}^{k} (-1)^{j-1} \binom{n}{j} \frac{\binom{3n-3j}{3k-3j}}{\binom{3n}{3k}}$$

by the same argument used in the solution to exercise 1.20. Note that this answer and solution (with obvious changes) generalizes to aliens with any number of legs. Unfortunately, the solution and answer of exercise 1.21, valid for earthlings with 2 legs, does **not** *generalize even to the case of 3 legs.*