TEST #2
STA 5326
October 30, 2008

Name: KEY

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.
- There are a total of **100** points.

Problem 1. A pdf $f(x)$ is said to be symmetric about a if $f(a+u) = f(a-u)$ for all u. Suppose the pdf of X is symmetric about a.

(a) (8%) Show that the median of X is a.

Proof: $\int_{-\infty}^{a} f(x)\,dx \overset{x-a=-u}{=} \int_{\infty}^{0} f(a-u)\,d(-u)$

$= \int_{0}^{\infty} f(a-u)\,du$

$\overset{f(a-u)=f(a+u)}{=\!=\!=} \int_{0}^{\infty} f(a+u)\,du$

$\overset{x-a=u}{=\!=\!=} \int_{a}^{\infty} f(x)\,dx$

And $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = 1$

$\therefore$ $\int_{-\infty}^{a} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx = \frac{1}{2}$ (definition of median)

$\therefore$ a is the median of X.

(b) (9%) Show that, if EX exists, then $EX = a$.

Proof: $\because EX = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{a} x f(x)\,dx + \int_{a}^{\infty} x f(x)\,dx$$

$$= \underbrace{\int_{\infty}^{0} (a-u) f(a-u)\,d(-u)}_{x-a=-u} + \underbrace{\int_{0}^{\infty} (a+u) f(a+u)\,du}_{x-a=u}$$

$$\overset{f(a-u)=f(a+u)}{=\!=} \int_{0}^{\infty} (a-u) f(a+u)\,du + \int_{0}^{\infty} (a+u) f(a+u)\,du$$

$$= 2\int_{0}^{\infty} a f(a+u)\,du + \int_{0}^{\infty} (u-u) f(a+u)\,du$$

$$\overset{x-a=u}{=\!=} 2a\int_{a}^{\infty} f(x)\,dx$$

Based on question (a), a is median of X

$$= a \cdot 2 \times \frac{1}{2} = a$$

Problem 2. Consider a sequence of independent trials, each of which has three possible outcomes: Red, Green, or Blue (abbreviated R, G, B), with probabilities p, q, r, respectively, where $p+q+r=1$. Let X be the length of the run (of either R, G, or B) started by the first trial. (For example, $X=3$ if you observe $RRRG$ or $BBBR$ or $GGGB$ or ...)

(a) (8%) What is $P(X=k)$ for $k=1,2,3,\ldots$?

R = the length of the run of R

G = the length of the run of G

B = the length of the run of B

$$P(X=k) = P\{(R=k) \cup (G=k) \cup (B=k)\} \quad R, G, B \text{ disjoint}$$

$$= P(R=k) + P(G=k) + P(B=k)$$

$$= p^k(1-p) + q^k(1-q) + r^k(1-r) \quad (k=1,2,3,\cdots)$$

(b) (9%) Find EX.

$$EX = \sum_{k=1}^{\infty} k[p^k(1-p) + q^k(1-q) + r^k(1-r)]$$

$$= \sum_{k=1}^{\infty} kp^k(1-p) + \sum_{k=1}^{\infty} kq^k(1-q) + \sum_{k=1}^{\infty} kr^k(1-r)$$

$$= p\sum_{k=1}^{\infty} k\underbrace{(1-p)p^{k-1}}_{\text{Geometric }(1-p)} + q\sum_{k=1}^{\infty} k\underbrace{(1-q)q^{k-1}}_{\text{Geometric }(1-q)} + r\sum_{k=1}^{\infty} k\underbrace{(1-r)r^{k-1}}_{\text{Geometric }(1-r)}$$

$$= \frac{p}{1-p} + \frac{q}{1-q} + \frac{r}{1-r}$$

Problem 3. (16%) Use indicator random variables to prove that

$P(A^c \cap B^c \cap C^c) = 1 - P(A) - P(B) - P(C) + P(A \cap B) + P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$.

Proof: $\because P(A^c \cap B^c C^c) = E(I_{A^c \cap B^c \cap C^c})$

And $I_{A^c \cap B^c \cap C^c} = I_{A^c} \cdot I_{B^c} \cdot I_{C^c}$

$= (1 - I_A)(1 - I_B)(1 - I_C)$

$= 1 - I_A - I_B - I_C + I_A \cdot I_B + I_A \cdot I_C + I_B \cdot I_C - I_A \cdot I_B \cdot I_C$

$= 1 - I_A - I_B - I_C + I_{A \cap B} + I_{A \cap C} + I_{B \cap C} - I_{A \cap B \cap C}$

$\therefore P(A^c \cap B^c \cap C^c) = E(1 - I_A - I_B - I_C + I_{A \cap B} + I_{A \cap C} + I_{B \cap C} - I_{A \cap B \cap C})$

$= E(1) - E(I_A) - E(I_B) - E(I_C) + E(I_{A \cap B}) + E(I_{A \cap C})$
$+ E(I_{B \cap C}) - E(I_{A \cap B \cap C})$

$= 1 - P(A) - P(B) - P(C) + P(A \cap B) + P(A \cap C) + P(B \cap C)$
$- P(A \cap B \cap C)$

Problem 4. (17%) Let $X \sim \text{Poisson}(\lambda = 225)$. Calculate a normal approximation for

$$P(203 \leq X \leq 238)\,.$$

$\because$ λ is large

$\therefore$ $X \sim$ approximate $N(\mu = 225, \sigma^2 = 225)$

And assume $D \sim$ exact $N(225, 225)$

$Z \sim N(0,1)$

$$\therefore P(203 \leq X \leq 238) \cong P(202.5 \leq D \leq 238.5)$$

$$= P\left(\frac{202.5-225}{\sqrt{225}} \leq \frac{D-225}{\sqrt{225}} \leq \frac{238.5-225}{\sqrt{225}}\right)$$

$$= P\left(\frac{202.5-225}{\sqrt{225}} \leq Z \leq \frac{238.5-225}{\sqrt{225}}\right)$$

$$= P(-1.5 \leq Z \leq 0.9)$$

$$= \Phi(0.9) - \Phi(-1.5)$$

$$= \Phi(0.9) - (1 - \Phi(1.5))$$

From Table

$$= 0.8159 - (1 - 0.9332)$$

$$= 0.7\cancel{4}491$$

Problem 5. A *0-truncated* discrete distribution is one in which the value 0 cannot be observed and is eliminated from the sample space. If X has range $\{0, 1, 2, \ldots\}$, the *0-truncated* random variable X_T has pmf

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)}, \quad x = 1, 2, 3, \ldots$$

Let X_T be the 0-truncated random variable starting from $X \sim \text{Binomial}(n, p)$.

(a) (5%) Find the pmf of X_T.

$$P(X>0) = 1 - P(X=0) = 1 - \binom{n}{0} p^0 (1-p)^n$$

$$= 1 - (1-p)^n$$

$$f_T(X_T) = P(X_T = x) = \frac{P(X=x)}{P(X>0)} = \frac{\binom{n}{x} p^x (1-p)^{n-x}}{1-(1-p)^n} \quad (x = 1, 2, 3, \ldots)$$

(b) (6%) Find EX_T.

$$EX_T = \sum_{x=1}^{n} x \frac{\binom{n}{x} p^x (1-p)^{n-x}}{1-(1-p)^n}$$

$$= \frac{1}{1-(1-p)^n} \underbrace{\sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}}_{np}$$

$$= \frac{np}{1-(1-p)^n}$$

(c) (6%) Find $\mathrm{Var}(X_T)$.

$$E X_T(X_T-1) = \sum_{x=1}^{n} x(x-1) \frac{\binom{n}{x} p^x (1-p)^{n-x}}{1-(1-p)^n}$$

$$= \frac{1}{1-(1-p)^n} \sum_{x=2}^{n} n(n-1)p^2 \binom{n-2}{x-2} p^{x-2} (1-p)^{(n-2)-(x-2)}$$

$$= \frac{n(n-1)p^2}{1-(1-p)^n} \sum_{x=2}^{n} \underbrace{\binom{n-2}{x-2} p^{x-2} (1-p)^{(n-2)-(x-2)}}_{1}$$

$$= \frac{n(n-1)p^2}{1-(1-p)^n}$$

$$\therefore \quad Var(X_T) = E X_T(X_T-1) + EX_T - (EX_T)^2$$

$$= \frac{n(n-1)p^2}{1-(1-p)^n} + \frac{np}{1-(1-p)^n} - \left(\frac{np}{1-(1-p)^n}\right)^2$$

$$= \frac{np(np-p+1)}{1-(1-p)^n} - \frac{n^2p^2}{(1-(1-p)^n)^2}$$

The remaining problems require no work. You will receive full credit for stating the correct answer.

Problem 6. (4%) Suppose $X \sim \text{Poisson}(\lambda = s)$ and $Y \sim \text{Gamma}(\alpha = k, \beta = 1)$ where $s > 0$, and k is a positive integer. The relationship

$$P(X < k) = P(Y > s)$$

may be written explicitly as a formula stating the equality of certain sum(s) and/or integral(s). Carefully state this formula, writing any sums or integrals explicitly.

$$\sum_{x=0}^{k-1} \frac{s^x e^{-s}}{x!} = \int_s^\infty \frac{y^{k-1} e^{-x}}{\Gamma(k)}$$

Problem 7. Suppose we observe (starting from time zero) a Poisson process with rate 3.

(a) (4%) What is the probability mass function of S_5, the number of arrivals during the interval $(0.0, 5.0)$? (Give an explicit formula.)

$$S_5 \sim \text{Poisson}(\lambda = 3 \times 5 = 15)$$

Pmf: $$P(S_5 = k) = \frac{e^{-15} 15^k}{k!} \quad (k = 0, 1, \ldots)$$

(b) (4%) What is the probability density function of T_7, the time of the 7^{th} arrival? (Give an explicit formula.)

$$T_7 \sim \text{Gamma}(\alpha = 7, \beta = \tfrac{1}{3})$$

$$f_T(t) = \frac{1}{\Gamma(7)(\frac{1}{3})^7} x^6 e^{-x/(\frac{1}{3})}$$

$$= \frac{3^7}{\Gamma(7)} x^6 e^{-3x} \quad (0 \le x < \infty)$$

Problem 8. (4%) An urn contains 3 red and 4 green balls which are randomly drawn out (one by one) without replacement until the urn is empty. What is the probability that the 3^{rd} and 5^{th} balls drawn are both red?

P{the 3rd and 5th balls drawn are both red}

= P{the first and second are red}

$$= \frac{3}{3+4} \cdot \frac{3-1}{3+4-1} = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$$