Problem 1. Suppose X and Y are independent with $X \sim \textbf{Uniform}(-1,1)$ and $Y \sim \textbf{Normal}(0,1)$. Define a new random variable Z by

$$Z = \begin{cases} X & \text{if } XY > 0 \\ -X & \text{if } XY < 0\,. \end{cases}$$

For $z > 0$

(a) (10%) What is the distribution of Z? Prove your answer.

Solutions: $P(Z > z) = P(X > z)P(XY > 0) + P(-X > z)P(XY < 0)$

$= P(X > z)\,P(Y > 0) + P(X < -z)\,P(Y > 0)$

Since $X \sim \text{Uniform}(-1,1)$ and $Y \sim \text{Normal}(0,1)$

$P(Z > z) = P(X > z)P(Y > 0) + P(X > z)\cdot P(Y \leq 0)$

$= P(X > z)$

We can also get $P(Z > z) = P(X > z)$ for $z \leq 0$

So $Z \sim X$, $Z \sim \text{Uniform}(-1, 1)$

(b) (4%) Are Z and Y independent? Prove your answer.

Solution: Z and Y are not independent

Since $Z = \begin{cases} X & \text{if } XY > 0 \\ -X & \text{if } XY < 0 \end{cases}$

when $Y > 0$, then $Z > 0$

when $Y < 0$, then $Z < 0$

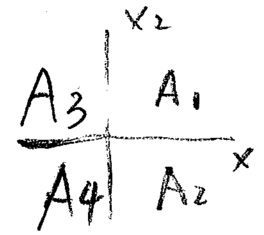
the sign of Z depends on Y.

So they are not independent

Problem 2. (20%) Suppose X_1 and X_2 are independent Normal$(0,\sigma^2)$ random variables. Find the joint density of

$$Y_1 = X_1^2 + X_2^2 \qquad \text{and} \qquad Y_2 = \frac{X_1^2}{X_1^2+X_2^2}.$$

solutions:



$$\begin{cases} Y_1 = X_1^2 + X_2^2 \\ Y_2 = \dfrac{X_1^2}{X_1^2+X_2^2} \end{cases} \longrightarrow \begin{cases} X_1^2 = Y_1 Y_2 \\ X_2^2 = Y_1 - Y_1 Y_2 \end{cases}$$

$A_1 = \{x_1, x_2 : x_1 > 0, x_2 > 0\}$, $A_2 = \{x_1, x_2 : x_1 > 0, x_2 < 0\}$

$A_3 = \{x_1, x_2 : x_1 < 0, x_2 > 0\}$, $A_4 = \{x_1, x_2 : x_1 < 0, x_2 < 0\}$

For A_1, $x_1 = \sqrt{y_1 y_2}$, $x_2 = \sqrt{y_1(1-y_2)}$

$$J_1 = \begin{vmatrix} \dfrac{\sqrt{y_2}}{2\sqrt{y_1}} & \dfrac{\sqrt{y_1}}{2\sqrt{y_2}} \\ \dfrac{1}{2}\sqrt{\dfrac{1-y_2}{y_1}} & -\dfrac{1}{2}\sqrt{\dfrac{y_1}{1-y_2}} \end{vmatrix} = -\frac{1}{4}\left[\sqrt{\frac{y_2}{1-y_2}} + \sqrt{\frac{1-y_2}{y_2}}\right] = -\frac{1}{4\sqrt{1-y_2}\cdot\sqrt{y_2}}$$

and $|J_1| = |J_2| = |J_3| = |J_4| = \dfrac{1}{4\sqrt{y_2(1-y_2)}}$

$$f_{Y_1,Y_2}(y_1,y_2) = \sum_{i=1}^{4} f_{X_1,X_2}\left(h_1(y), h_i(y)\right)\cdot |J_i|$$

$$= \frac{1}{\sqrt{y_2(1-y_2)}}\cdot\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y_1 y_2}{2\sigma^2}}\cdot\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y_1(1-y_2)}{2\sigma^2}}$$

$$= \frac{1}{2\pi\sigma^2}\,\frac{1}{\sqrt{y_2(1-y_2)}}\, e^{-\frac{y_1}{2\sigma^2}}$$

where $y_1 > 0$ and $0 \le y_2 \le 1$

Problem 3. The random pair (X, Y) has the distribution (joint mass function)

		X			
		1	2	3	
	1	1/6	1/12	1/12	$\frac{1}{3}$
Y	2	1/12	1/6	1/12	$\frac{1}{3}$
	3	1/12	1/12	1/6	$\frac{1}{3}$
		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

(a) (7%) Show that X and Y are dependent.

Solution: $p(X=1) = p(Y=1, X=1) + p(Y=2, X=1) + p(Y=3, X=1) = \frac{1}{3}$

$p(Y=1) = p(X=1, Y=1) + p(X=2, Y=1) + p(X=3, Y=1) = \frac{1}{3}$

$p(X=1, Y=1) = \frac{1}{6}$

$p(X=1, Y=1) \neq p(X=1) \cdot p(Y=1)$

So X and Y are dependent.

(b) (7%) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

Solution: We get $p(U=1) = p(U=2) = p(U=3) = \frac{1}{3}$

$p(V=1) = p(V=2) = p(V=3) = \frac{1}{3}$

		U		
		1	2	3
	1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
V	2	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
	3	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

U and V are indep where

$P(U=i, V=j) = P(U=i)P(V=j)$

$i = 1, 2, 3$

$j = 1, 2, 3$

Problem 4.

(a) (6%) State the definition of a two-parameter exponential family (2pef), that is, give a general expression for the density function of a 2pef.

$$f(x) = h(x)\, c(\theta)\, \exp\left[\sum_{i=1}^{2} w_i(\theta)\, t_i(x)\right]$$

(b) (6%) Does the Gamma(α, β) family with both α and β unknown form a 2pef? Justify your answer.

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \quad 0 \le x < \infty.$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{-x/\beta} \cdot e^{(\alpha-1)\log x}, \quad \theta = (\alpha, \beta)$$

$$= \underbrace{I_{(0,\infty)}(x)}_{h(x)} \cdot \underbrace{\frac{1}{\Gamma(\alpha)\beta^{\alpha}}}_{c(\theta)} \exp\left[-\underbrace{\frac{1}{\beta}}_{W_1(\theta)} \cdot \underbrace{x}_{t_1(x)} + \underbrace{(\alpha-1)}_{W_2(\theta)} \cdot \underbrace{\log x}_{t_2(x)} \right]$$

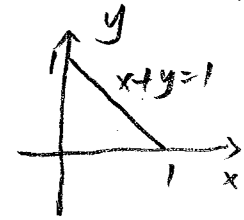
So Gamma(α, β) family with α and β unknown forms a 2pef.

(c) (5%) Does the Uniform(a, b) family with both a and b unknown form a 2pef? Justify your answer.

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b \qquad \theta = (a, b)$$

Since the support $\{x : a \le x \le b\}$ depends on the parameter $\theta = (a, b)$, so the Uniform(a, b) doesn't form a 2pef.

Problem 5. Suppose X, Y have joint density $f_{X,Y}(x,y) = 6(1-x-y)$ in the region where $x > 0$, $y > 0$, and $x + y < 1$ (and zero outside this region).



(a) (5%) Find $f_X(x)$.

$$f_X(x) = \int_0^{1-x} 6(1-x-y)\,dy$$

$$= -3(1-x-y)^2\Big|_0^{1-x}$$

$$= 3(1-x)^2 \qquad 0<x<1$$

(b) (5%) Find $f_{Y|X}(y|x)$ for $0 < x < 1$.

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{6(1-x-y)}{3(1-x)^2} = \frac{2(1-x-y)}{(1-x)^2}$$

where $0<x<1$ and $0<y<1-x$.

(c) (5%) Find $E(Y|X = x)$ for $0 < x < 1$.

$$E(Y|X=x) = \int_0^{1-x} y \cdot f(y|x)\,dy$$

$$= \int_0^{1-x} \frac{2y(1-x-y)}{(1-x)^2}\,dy$$

$$= \frac{1}{(1-x)^2}\left[(1-x)y^2 - \frac{2}{3}y^3\right]\Big|_0^{1-x}$$

$$= \frac{1-x}{3} \quad \text{for} \quad 0<x<1$$

No work is required in the remaining problems. You will receive full credit for stating the correct answers.

Problem 6.

(a) (4%) State a general expression for $\mathrm{Var}(Y|X)$ as a difference of two terms.

$\mathbf{Var}(\boldsymbol{Y}|\boldsymbol{X}) =$ $E(Y^2|X) - [E(Y|X)]^2$

(b) (4%) Use the answer to part (a) and obtain an expression for $E[\mathbf{Var}(Y|X)]$. Simplify where possible.

$\boldsymbol{E}[\mathbf{Var}(\boldsymbol{Y}|\boldsymbol{X})] =$ $EY^2 - E\{[E(Y|X)]^2\}$

(c) (4%) Obtain an expression for $\mathrm{Var}[E(Y|X)]$ as a difference of two terms. Simplify where possible.

$\mathbf{Var}[\boldsymbol{E}(\boldsymbol{Y}|\boldsymbol{X})] =$ $E\{[E(Y|X)]^2\} - (EY)^2$

Problem 7. Suppose $X_1, X_2, \ldots, X_n$ are iid from a distribution with density f and cdf F.

(a) (4%) State a formula for the cdf of $X_{(j)}$, the j-th order statistic.

$$F_{X(j)}(x) = \sum_{k=j}^{n} \binom{n}{k} [F(x)]^k [1-F(x)]^{n-k}$$

(b) (4%) State a formula for the density of $X_{(j)}$.

$$f_{X(j)}(x) = n \binom{n-1}{j-1} f(x) \cdot [F(x)]^{j-1} \cdot [1-F(x)]^{n-j}$$