TEST #3
STA 5326 **Name:**______________________________________
December 4, 2008

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- **If your answer is valid only in a certain range, this should be stated as part of the answer.**
- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- Show and explain your work (including your calculations) for all the problems except those on the last two pages. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** pages.
- There are a total of **100** points.

Problem 1. Suppose X and Y are independent with $X \sim$ **Uniform**$(-1, 1)$ and $Y \sim$ **Normal**$(0, 1)$. Define a new random variable Z by

$$Z = \begin{cases} X & \text{if } XY > 0 \\ -X & \text{if } XY < 0 \,. \end{cases}$$

This problem is similar to exercise 4.47

(a) (10%) What is the distribution of Z? Prove your answer.

$Z \sim$ *Uniform*$(-1, 1)$. *The solution to 4.47(a) works without change to show that Z has the same distribution as X.*

(b) (4%) Are Z and Y independent? Prove your answer.

They are **not** *independent. The definition of Z forces Z and Y to have the same sign. Thus*

$$0 = P(Z > 0, Y < 0) \neq P(Z > 0)P(Y < 0) = (1/2) \cdot (1/2) = 1/4 \,.$$

Give students full credit for stating "No, because Z and Y have the same sign."

Problem 2. (20%) Suppose X_1 and X_2 are independent Normal$(0, \sigma^2)$ random variables. Find the joint density of

$$Y_1 = X_1^2 + X_2^2 \qquad \text{and} \qquad Y_2 = \frac{X_1^2}{X_1^2 + X_2^2}\,.$$

This is a modification of exercise 4.20. The bivariate transformation is 4-to-1, but is 1-to-1 when restricted to each of the quadrants. The answer is a sum of four terms (one from each quadrant) which are all identical, leading to a factor of 4 in the final answer. It is not necessary to work out all four cases. If a student says there are four cases, each with an identical contribution, and then works out one case in detail, and gets the correct answer (including the factor of 4), they should receive full credit.

Another approach is to find the density of $Z_i = X_i^2$, $i = 1, 2$, *and then transform from* Z_1, Z_2 *to* $Y_1 = Z_1 + Z_2$ *and* $Y_2 = Z_1/(Z_1 + Z_2)$.

Problem 3. The random pair (X, Y) has the distribution (joint mass function)

		X		
		1	2	3
	1	1/6	1/12	1/12
Y	2	1/12	1/6	1/12
	3	1/12	1/12	1/6

This is similar to exercise 4.10.

(a) (7%) Show that X and Y are dependent.

(b) (7%) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

Problem 4.

(a) (6%) State the definition of a two-parameter exponential family (2pef), that is, give a general expression for the density function of a 2pef.

(b) (6%) Does the Gamma(α, β) family with both α and β unknown form a 2pef? Justify your answer.

The answer is "Yes.". This is part of exercise 3.28.

(c) (5%) Does the Uniform(a, b) family with both a and b unknown form a 2pef? Justify your answer.

No, because the support (a, b) depends on the parameters. Stating this is sufficient for full credit.

Problem 5. Suppose X, Y have joint density $f_{X,Y}(x, y) = 6(1-x-y)$ in the region where $x > 0$, $y > 0$, and $x + y < 1$ (and zero outside this region).

This is taken from notes10.pdf. See pages 9, 10, and 14.

(a) (5%) Find $f_X(x)$.

(b) (5%) Find $f_{Y|X}(y|x)$ for $0 < x < 1$.

(c) (5%) Find $E(Y|X = x)$ for $0 < x < 1$.

No work is required in the remaining problems. You will receive full credit for stating the correct answers.

Problem 6.

This problem is based on page 3 of notes12.pdf.

(a) (4%) State a general expression for $\text{Var}(Y|X)$ as a difference of two terms.

$\textbf{Var}(\boldsymbol{Y}|\boldsymbol{X}) =$

(b) (4%) Use the answer to part (a) and obtain an expression for $E[\textbf{Var}(Y|X)]$. Simplify where possible.

$\boldsymbol{E}[\textbf{Var}(\boldsymbol{Y}|\boldsymbol{X})] =$

(c) (4%) Obtain an expression for $\text{Var}[E(Y|X)]$ as a difference of two terms. Simplify where possible.

$\textbf{Var}[\boldsymbol{E}(\boldsymbol{Y}|\boldsymbol{X})] =$

Problem 7. Suppose $X_1, X_2, \ldots, X_n$ are iid from a distribution with density f and cdf F. *See notes14.pdf.*

(a) (4%) State a formula for the cdf of $X_{(j)}$, the j-th order statistic.

(b) (4%) State a formula for the density of $X_{(j)}$.