**TEST #1**
**STA 5326** **Name:**____________________________________
**September 24, 2009**

**Please read the following directions.**
**DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO**

# Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** pages.
- There are a total of **100** points.

**Problem 1.** (18%) Suppose you have a fair coin with the sides labeled +1 and −1. Toss this coin 3 times and let $X_i$ be the value observed on the $i$-th toss. Define $X_4 = X_1X_2X_3$. For $i = 1, 2, 3, 4$, define $A_i$ to be the event that $X_i = 1$.

Show that the events $A_1, A_2, A_3, A_4$ are *not* mutually independent.

$A_1 = \{X_1 = 1\}$, $A_2 = \{X_2 = 1\}$, $A_3 = \{X_3 = 1\}$, $A_4 = \{X_4 = 1\}$

Here $X_4 = X_1X_2X_3$; $X_1, X_2, X_3$ are independent, $X_i = \{\pm 1\}$.

If $A_1, A_2, A_3, A_4$ are mutually independent, then

$P(A_1 \cap A_2) = P(A_1)P(A_2)$; $P(A_1 \cap A_3) = P(A_1)P(A_3)$;

$P(A_1 \cap A_4) = P(A_1)P(A_4)$; $P(A_2 \cap A_3) = P(A_2)P(A_3)$;

$P(A_2 \cap A_4) = P(A_2)P(A_4)$; $P(A_3 \cap A_4) = P(A_3)P(A_4)$;

$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$; $P(A_1 \cap A_2 \cap A_4) = P(A_1)P(A_2)P(A_4)$

$P(A_1 \cap A_3 \cap A_4) = P(A_1)P(A_3)P(A_4)$; $P(A_2 \cap A_3 \cap A_4) = P(A_2)P(A_3)P(A_4)$

$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)P(A_2)P(A_3)P(A_4)$.

But, for this problem: $P(A_1) = 1/2 = P(\{X_1 = 1\})$

$P(A_2) = 1/2 = P(\{X_2 = 1\})$

$P(A_3) = 1/2 = P(\{X_3 = 1\})$

$P(A_4) = P(\{X_4 = 1\}) = P(\{X_1X_2X_3 = 1\})$

$= P(\{X_1 = X_2 = X_3 = 1;\ X_1 = X_2 = -1 \,\&\, X_3 = 1;\ X_1 = X_3 = -1 \,\&\, X_2 = 1;$
$X_2 = X_3 = -1 \,\&\, X_1 = 1\})$

$= 4 * (\frac{1}{2})^3 = 1/2$

$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(\{X_1 = 1, X_2 = 1, X_3 = 1, X_4 = 1\}) = P(\{X_1 = 1 = X_2 = X_3,\ X_1X_2X_3 = 1\})$

$= (\frac{1}{2})^3 = 1/8 \neq P(A_1)P(A_2)P(A_3)P(A_4) = \frac{1}{16}$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $X_1$ | 1 | 1 | 1 | 1 | −1 | −1 | −1 | −1 |
| $X_2$ | 1 | 1 | −1 | −1 | 1 | +1 | −1 | −1 |
| $X_3$ | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |
| $X_4$ | 1 | −1 | −1 | 1 | −1 | 1 | 1 | −1 |

So, the events $A_1, A_2, A_3, A_4$ are not mutually independent.

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**Problem 2.** Let $X \sim \text{Poisson}(1)$. Calculate the following.

**(a)** (9%) $P(X \geq 3)$

$$P(X \geq 3) = 1 - P(X < 3) = 1 - P(X=2) - P(X=1) - P(X=0)$$

$$= 1 - \frac{e^{-\lambda}\lambda^2}{2!} - \frac{e^{-\lambda}\lambda^1}{1!} - \frac{e^{-\lambda}\lambda^0}{0!} \qquad \lambda = 1 \text{ here}$$

$$= 1 - \frac{e^{-1}}{2} - \frac{e^{-1}}{1} - \frac{e^{-1}}{1}$$

$$= 1 - \frac{5}{2}e^{-1}$$

**(b)** (9%) $E\left(\frac{1}{(X+1)(X+2)}\right)$

$$P(X=x) = \frac{e^{-1}}{x!}$$

$$E\left(\frac{1}{(X+1)(X+2)}\right) = \sum_{x=0}^{+\infty} \frac{1}{(x+1)(x+2)} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{+\infty} \frac{e^{-\lambda}\lambda^x}{(x+2)!} = \sum_{x=0}^{+\infty} \frac{e^{-\lambda}\lambda^{x+2}}{(x+2)!} \frac{1}{\lambda^2}$$

let $y = x+2$

$$= \left[\sum_{y=0}^{+\infty} \frac{e^{-\lambda}\lambda^y}{y!} - \frac{e^{-\lambda}\lambda^0}{0!} - \frac{e^{-\lambda}\lambda^1}{1!}\right] \frac{1}{\lambda^2}$$

$$= (1 - e^{-\lambda} - e^{-\lambda}) \frac{1}{\lambda^2} \qquad \lambda = 1$$

$$= 1 - \frac{2}{e}$$

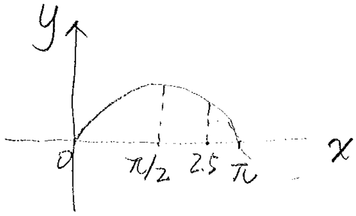
$-0$

**Problem 3.** (18%) Suppose $X$ has density (pdf)

$$f_X(x) = .32\,x \quad \text{for } 0 < x < 2.5$$

Find the density (pdf) of $Y = \sin(X)$.

[ Some possibly useful facts: $\frac{d}{dy}\left(\sin^{-1}(y)\right) = \frac{1}{\sqrt{1-y^2}}$ and $\sin(2.5) = 0.5984721$. ]




$Y = \sin(x)$. $\quad f_X(x) = .32x \geq 0$

$g_1(x) = \sin x$, $\quad x \in (0, \pi/2] = A_1$

$g_2(x) = \sin x$, $\quad x \in (\frac{\pi}{2}, 2.5) = A_2$

$g_1^{-1}(y) = \sin^{-1}(y)$ $\quad y \in (0, 1] = B_1$

$\frac{d g_1^{-1}(y)}{dy} = \frac{1}{\sqrt{1-y^2}}$, $\quad g_1(x)$ increasing at $A_1$

$g_2^{-1}(y) = -\sin^{-1}(y) + \pi$, $\quad y \in (0.5984721, 1) = B_2$

$\frac{d g_2^{-1}(y)}{dy} = -\frac{1}{\sqrt{1-y^2}}$, $\quad g_2(x)$ decreasing at $A_2$

As $X$ has a pdf $f_X(x) = .32x$ for $0 < x < 2.5$, and $g_1(x)$ increasing at $A_1$, $g_2(x)$ decreasing at $A_2$.

$$f_Y(y) = \sum_{i=1}^{2} f_X(g_i^{-1}(y)) \left| \frac{d g_i^{-1}(y)}{dy} \right| I_{B_i}(y)$$

$$= .32 \sin^{-1}(y) \cdot \frac{1}{\sqrt{1-y^2}} I_{(0,1]}(y) + .32(\pi - \sin^{-1}(y)) \frac{1}{\sqrt{1-y^2}} I_{(.5984721, 1)}(y)$$

$$= \begin{cases} .32 * \frac{\sin^{-1}(y)}{\sqrt{1-y^2}} & 0 < y \leq .5984721 \\ .32 * \frac{\pi}{\sqrt{1-y^2}} & .5984721 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

but OK

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**Problem 4.** A monkey types **5** letters at random. (Each keystroke is independent of the others with all 26 possibilities equally likely.)

**(a)** (9%) What is the probability the monkey types OGRE? (That is, the letters OGRE occur as four consecutive letters somewhere in the 5 typed letters.)

$A_1$: OGRE?

$A_2$: ?OGRE

$A_1 \cap A_2 = \phi$

$P(\text{the monkey types OGRE}) = P(A_1 \cup A_2)$

$= P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$P(A_1) = (\frac{1}{26})^4$, $P(A_2) = (\frac{1}{26})^4$

so $P(\text{the monkey types OGRE}) = (\frac{1}{26})^4 + (\frac{1}{26})^4 - 0$

$= 2 * (\frac{1}{26})^4$

**(b)** (9%) What is the probability the monkey types BIT or TAB?

$a_1$: BIT?? $\quad$ $a_4$: TAB??

$a_2$: ?BIT? $\quad$ $a_5$: ?TAB?

$a_3$: ??BIT $\quad$ $a_6$: ??TAB

$P(\text{the monkey types BIT or TAB}) = P(a_1 \cup a_2 \cup a_3 \cup a_4 \cup a_5 \cup a_6)$

$P(a_i) = (\frac{1}{26})^3$, $\quad a_1 \cap a_2 = \phi$, $a_1 \cap a_3 = \phi$ $a_1 \cap a_4 = \phi$. $a_1 \cap a_5 = \phi$,

$a_1 \cap a_6 = \text{BITAB}$ $\quad P(a_1 \cap a_6) = (\frac{1}{26})^5$

$a_2 \cap a_3 = \phi$, $a_2 \cap a_4 = \phi$, $a_2 \cap a_5 = \phi$. $a_2 \cap a_6 = \phi$

$a_3 \cap a_4 = \text{TABIT}$ $\quad a_3 \cap a_5 = \phi$. $a_3 \cap a_6 = \phi$

$a_4 \cap a_5 = \phi$. $a_4 \cap a_6 = \phi$; $a_5 \cap a_6 = \phi$

Any 3 of the sets' intersection is empty

Any 4 of the sets' intersection is empty

Any 5 of the sets intersection is empty

$a_1 \cap a_2 \cap a_3 \cap a_4 \cap a_5 \cap a_6 = \phi$.

$P(a_1 \cup a_2 \cup a_3 \cup a_4 \cup a_5 \cup a_6) = \sum_{i=1}^{6} P(a_i) - P(a_1 \cap a_6) - P(a_3 \cap a_4)$

$= 6 * (\frac{1}{26})^3 - 2 * (\frac{1}{26})^5$

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**Problem 5.** (9%) **Seven** cards are dealt from a well-shuffled standard deck of 52 cards. What is the probability these **seven** cards contain exactly two hearts and two diamonds?

52 cards have 13 diamonds and 13 hearts

$\Omega$ : $\binom{52}{7}$ kinds

A : seven cards contain exactly two hearts and two diamonds

$\binom{13}{2}\cdot\binom{13}{2}\cdot\binom{26}{3}$

$$P = \frac{\#(A)}{\#(\Omega)} = \frac{\binom{13}{2}\binom{13}{2}\binom{26}{3}}{\binom{52}{7}}$$

$$= \frac{\frac{13!}{2!11!}\ \frac{13!}{2!11!}\ \frac{26!}{3!23!}}{\frac{52!}{7!45!}}$$

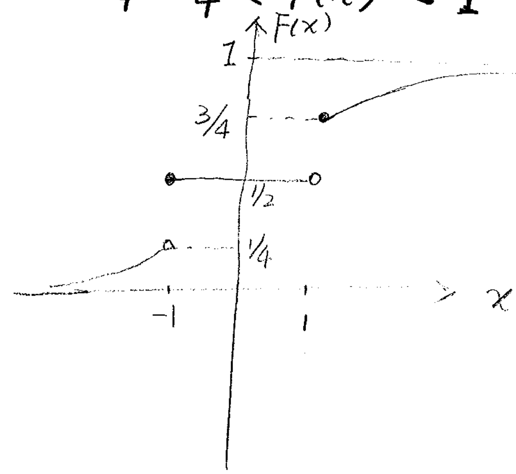
**Problem 6.** (8%) Consider the cdf

$$F(x) = \begin{cases} \frac{1}{4x^2} & x < -1 \\ \frac{1}{2} & -1 \le x < 1 \\ 1 - \frac{1}{4x^2} & x \ge 1 \end{cases}$$

Find $F^{-1}(y)$ where $F^{-1}(y)$ is defined for $0 < y < 1$ by

$$F^{-1}(y) = \inf\{x : F(x) \ge y\}\,.$$

$$F(x) = \begin{cases} \frac{1}{4x^2} & x < -1 & \Rightarrow 0 < F(x) < \frac{1}{4}, & \text{increasing} \\ \frac{1}{2} & -1 \le x < 1 & \Rightarrow F(x) = \frac{1}{2}, & \text{flat} \\ 1 - \frac{1}{4x^2} & x \ge 1 & \Rightarrow \cancel{0 < F(x) \le \frac{3}{4}}\ \frac{3}{4} \le F(x) < 1, & \text{increasing} \end{cases}$$




$F^{-1}(y) = \inf\{x : F(x) \ge y\}$.

For $0 < y < \frac{1}{4}$

$y = \frac{1}{4x^2} \Rightarrow x = -\frac{1}{2\sqrt{y}} = F^{-1}(y)$.

For $y = 1/2$

$F^{-1}(y) = -1$

For $3/4 \le y < 1$

$y = 1 - \frac{1}{4x^2} \Rightarrow x = \frac{1}{2\sqrt{1-y}} = F^{-1}(y)$.

so $F^{-1}(y) = \begin{cases} -\frac{1}{2\sqrt{y}} & 0 < y < \frac{1}{4} \\ -1 & y = 1/2 \\ \frac{1}{2\sqrt{1-y}} & \frac{3}{4} \le y < 1 \end{cases}$

This is okay, but these two can be combined as one statement.

For $\frac{1}{4} \le y < 1/2$, $F^{-1}(y) = -1$

For $\frac{1}{2} < y < \frac{3}{4}$, $F^{-1}(y) = +1$

-0

**Problem 7.** (3%) A biased coin with probability 2/3 of heads is tossed 5 times and the observed sequence of heads and tails is recorded. For this experiment, how many outcomes $\omega$ are there in the sample space $\Omega$? (Fill in the blank.)

Number of outcomes = $2^5$

**Problem 8.** (4%) Which of the following are properties of all cumulative distribution functions (cdf's)? (Circle all correct responses.)

a) $F(\infty) = 1$

b) $F(\infty) = 0$

c) continuity

d) $F(-\infty) = 0$

e) $F(-\infty) = 1$

f) left continuity

g) right continuity

h) $F$ integrates to 1

i) $F$ is nonincreasing

j) $F$ is nondecreasing

k) $F$ is strictly increasing

**Problem 9.** (4%) Which of the following statements are **NOT** always true. (Circle all of them.)

a) $P(\emptyset) = 0$

b) $P(\Omega) = 1$

c) $P(A^c) = 1 - P(A)$

d) $P(\cup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} P(A_i)$.

e) If $A \subset B$, then $P(A) < P(B)$. If $P(A)=0$, $P(B)=0$ then $P(A)=P(B)$ though $A \subset B$

f) $P(B \cap A^c) = P(B) - P(A \cap B)$

g) $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$ for any partition $C_1, C_2, C_3, \ldots$

h) $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ when $A_1, A_2, A_3, \ldots$ are disjoint.

i) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

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