TEST #1
STA 5326 **Name:**___________________________________
September 24, 2009

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** pages.
- There are a total of **100** points.

Problem 1. (18%) Suppose you have a fair coin with the sides labeled $+1$ and -1. Toss this coin 3 times and let X_i be the value observed on the i-th toss. Define $X_4 = X_1X_2X_3$. For $i = 1, 2, 3, 4$, define A_i to be the event that $X_i = 1$.

Show that the events A_1, A_2, A_3, A_4 are *not* mutually independent.

This is identical to exercise B8, part (b)

Problem 2. Let $X \sim \text{Poisson}(1)$. Calculate the following.

This is a slight modification of exercise A1. The summation for the expected value in part (b) has a simple closed form.

(a) (9%) $P(X \geq 3)$

(b) (9%) $E\left(\dfrac{1}{(X+1)(X+2)}\right)$

Problem 3. (18%) Suppose X has density (pdf)

$$f_X(x) = .32\,x \quad \text{for } 0 < x < 2.5$$

Find the density (pdf) of $Y = \sin(X)$.

[Some possibly useful facts: $\dfrac{d}{dy}\left(\sin^{-1}(y)\right) = \dfrac{1}{\sqrt{1-y^2}}$ and $\sin(2.5) = 0.5984721$.]

This is a modification of an example from lecture on pages 22-25 of notes3.pdf. There are two different solutions in lecture, and both approaches can be used in this problem. Or one can use the Variation of Theorem 2.1.8 given on page 26 of notes3.pdf. The pdf of Y takes different forms in the intervals $(0, \sin(2.5))$ and $(\sin(2.5), 1)$. In the first interval the pdf consists of a single term, and in the second it is a sum of two terms. Outside these two intervals the pdf is zero.

Problem 4. A monkey types **5** letters at random. (Each keystroke is independent of the others with all 26 possibilities equally likely.)

This is a modification of exercise B1 and is similar to some examples from lecture on pages 11-14 of notes1.pdf.

(a) (9%) What is the probability the monkey types OGRE? (That is, the letters OGRE occur as four consecutive letters somewhere in the **5** typed letters.)

(b) (9%) What is the probability the monkey types BIT or TAB?

This part is similar to part (b) of B1, but there is an important difference: BIT and TAB can overlap as BITAB or TABIT. The event "types BIT or TAB" can be written as a union of 6 events" and the probability obtained by inclusion-exclusion. Most of the intersections are empty, but not all (because of the possibility of BIT and TAB overlapping).

Problem 5. (9%) **Seven** cards are dealt from a well-shuffled standard deck of 52 cards. What is the probability these **seven** cards contain exactly two hearts and two diamonds?

This is somewhat similar to (but simpler than) exercise 1.22 or the full-house poker problem in notes1.pdf (see pages 16 and 24-25).

Problem 6. (8%) Consider the cdf

$$F(x) = \begin{cases} \frac{1}{4x^2} & x < -1 \\ \frac{1}{2} & -1 \le x < 1 \\ 1 - \frac{1}{4x^2} & x \ge 1 \end{cases}$$

Find $F^{-1}(y)$ where $F^{-1}(y)$ is defined for $0 < y < 1$ by

$$F^{-1}(y) = \inf\{x : F(x) \ge y\}\,.$$

This problem is similar to exercise 2.8, and combines features of parts (b) and (c) of that problem.

Problem 7. (3%) A biased coin with probability 2/3 of heads is tossed 5 times and the observed sequence of heads and tails is recorded. For this experiment, how many outcomes ω are there in the sample space Ω? (Fill in the blank.)

Number of outcomes = ________________

Problem 8. (4%) Which of the following are properties of all cumulative distribution functions (cdf's)? (Circle all correct responses.)

a) $F(\infty) = 1$

b) $F(\infty) = 0$

c) continuity

d) $F(-\infty) = 0$

e) $F(-\infty) = 1$

f) left continuity

g) right continuity

h) F integrates to 1

i) F is nonincreasing

j) F is nondecreasing

k) F is strictly increasing

Problem 9. (4%) Which of the following statements are **NOT** always true. (Circle all of them.)

a) $P(\emptyset) = 0$

b) $P(\Omega) = 1$

c) $P(A^c) = 1 - P(A)$

d) $P(\cup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} P(A_i)$.

e) If $A \subset B$, then $P(A) < P(B)$.

f) $P(B \cap A^c) = P(B) - P(A \cap B)$

g) $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$ for any partition $C_1, C_2, C_3, \ldots$

h) $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ when $A_1, A_2, A_3, \ldots$ are disjoint.

i) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$