TEST #2
STA 5326 **Name:**____________________________________
October 29, 2009

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely **unless numerical answers are requested**. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.
- There are a total of **100** points.

Problem 1. (15%) Suppose that X is a continuous random variable with median m, that is, $P(X \le m) = P(X \ge m) = 1/2$. Show that $E|X-a|$ is minimized when $a = m$.

$$E|X-a| = \int_{-\infty}^{+\infty} |X-a| f(x)\,dx$$

$$= \int_{a}^{+\infty} (x-a) f(x)\,dx + \int_{-\infty}^{a} (a-x) f(x)\,dx$$

$$\int_{-\infty}^{m} f(x)\,dx = \tfrac{1}{2} = \int_{m}^{+\infty} f(x)\,dx \,, \quad m \text{ is the median}$$

$E|X-a|$ is a continuous function of a. and it is minimized when $\dfrac{dE|X-a|}{da} = 0$ and $\dfrac{d^2E|X-a|}{da^2} > 0$

$$\frac{dE|X-a|}{da} = -1 * (a-a) f(a) + \int_{a}^{+\infty} (-1) f(x)\,dx + 1 * (a-a) f(a) + \int_{-\infty}^{a} 1 * f(x)\,dx$$

$$= \int_{a}^{+\infty} (-f(x))\,dx + \int_{-\infty}^{a} f(x)\,dx = 0$$

$$\Rightarrow \int_{a}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx \quad \Rightarrow a \text{ is the median}$$

$$\left[\int_{a}^{+\infty} f(x)\,dx + \int_{-\infty}^{a} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{a} f\,dx = \int_{a}^{+\infty} f\,dx = \tfrac{1}{2} \right].$$

$$\frac{d^2E|X-a|}{da^2} = -1(-f(a)) + 1 * f(a) = 2f(a)\Big|_{a=m} \ge 0$$

m is the only stationary point for $E|X-a|$, and $2f(m) \ge 0$

so $E|X-a|$ is minimized when $a = m$.

-0

Problem 2. An urn initially contains two balls, one red and one green. This is a Polya urn: after a ball is drawn, it is replaced and then another ball of the same color is added to the urn. Ed decides to randomly draw balls until he finally gets a green ball. Then he will stop. Let X be the total number of balls that Ed draws from the urn.

(a) (8%) For $k = 1, 2, 3, \ldots$, what is $P(X = k)$?

$P(X=k)$ = probability { draw ~~green~~ red balls first (k-1) times, at kth time draw ~~green~~ green ball }.

$= p$ { draw red ball at time 1, 2, ⋯, k-1, and greenball at k }

$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} * \cdots * \frac{k-2}{k-1} * \frac{k-1}{k} * \frac{1}{k+1}$

$= \frac{1}{k(k+1)} \qquad k = 1, 2, 3, \cdots$

(b) (7%) What is EX?

$EX = \sum_{k=1}^{\infty} k\, p(X=k) = \sum_{k=1}^{\infty} k * \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k+1}$

$= \frac{1}{2} + \frac{1}{3} + \cdots + \cdots = +\infty$

-0

Problem 3. Let X have the density (pdf)

$$f(x) = cx^3 e^{-x^2/\beta^2}, \quad 0 < x < \infty, \quad \beta > 0.$$

(a) (7%) Find the value of the normalizing constant c. (c will depend on β.)

$f(x) = cx^3 e^{-x^2/\beta^2}$ is the pdf for $0 < x < \infty$, $\beta > 0$

so $f(x) \geq 0 \Rightarrow c > 0$ ($c \neq 0$, ortherwise $f \equiv 0$).

Then, $\int_0^\infty f(x)\,dx = 1$

$$\Rightarrow \int_0^\infty cx^3 e^{-x^2/\beta^2} dx = 1 \Rightarrow \frac{\beta^2 c}{-2} \int_0^\infty x^2 \, d e^{-x^2/\beta^2} = 1$$

$$\Rightarrow \frac{\beta^2 c}{-2}\left[x^2 e^{-x^2/\beta^2}\Big|_0^\infty - \int_0^\infty e^{-x^2/\beta^2} \cdot 2x\,dx\right] = 1$$

$$\Rightarrow \frac{\beta^2 c}{+2} \int_0^\infty 2x e^{-x^2/\beta^2} dx = 1 \Rightarrow \frac{\beta^2 c}{2} \int_0^\infty d e^{-x^2/\beta^2} \cdot (-\beta^2) = 1$$

$$\Rightarrow -\frac{c\beta^4}{2} e^{-x^2/\beta^2}\Big|_0^\infty = 1 \Rightarrow \frac{c\beta^4}{2} = 1 \Rightarrow c = \frac{2}{\beta^4}$$

(b) (8%) Find EX.
[If you could not find the value of c, just leave it as c in your answer.]

EX, First check $E|X|$. as $f(x)$ is defined on $0<x<\infty$

so $E|X| = EX$, Just need to calculate EX

$$EX = \int_0^{+\infty} x\, cx^3 e^{-x^2/\beta^2} dx.$$

$$= c\int_0^\infty x^4 e^{-x^2/\beta^2} dx \qquad \text{let } x = \sqrt{t} \Rightarrow dx = \tfrac{1}{2} t^{-1/2} dt$$

$$= c\int_0^\infty t^2 e^{-t/\beta^2} \cdot \tfrac{1}{2} t^{-1/2} dt$$

$$= \frac{c}{2}\int_0^\infty t^{3/2} e^{-t/\beta^2} dt = \frac{c}{2}\Gamma(\tilde{\alpha})(\tilde{\beta})^{\tilde{\alpha}} \underbrace{\int_0^\infty \frac{t^{5/2-1} e^{-t/\beta^2}}{\Gamma(\frac{5}{2})(\beta^2)^{5/2}} dt}_{1}$$

$\tilde{\alpha} = 5/2$, $\tilde{\beta} = \beta^2$

$$\Rightarrow EX = \frac{c}{2}\Gamma(\tfrac{5}{2})(\beta^2)^{5/2} = \frac{2}{\beta^4} * \frac{1}{2} * \frac{3}{2} * \frac{1}{2}\Gamma(\tfrac{1}{2}) * \beta^5$$

$$= \frac{3}{4}\beta * \Gamma(\tfrac{1}{2}) = \frac{3}{4}\beta\sqrt{\pi}.$$

4

-0

Problem 4. Suppose X and Y are iid random variables with moment generating function

$$M_X(t) = M_Y(t) = \frac{1}{(1+t)(1-t)} \quad \text{for } -1 < t < 1 \text{ (and undefined or } +\infty \text{ otherwise).}$$

Find the following moment generating functions. In each case, your answer should include the range in which the mgf is well-defined and finite.

(a) (8%) Find the mgf of $3X + 5$.

$$M_{3X+5}(t) = M_{3X}(t)\, e^{5t} = M_X(3t)\, e^{5t}, \quad -1 < 3t < 1$$

$$= \frac{1}{(1+3t)(1-3t)} * e^{5t} = \frac{e^{5t}}{(1+3t)(1-3t)}$$

range of t: $-1 < 3t < 1 \Rightarrow -\frac{1}{3} < t < \frac{1}{3}$

so $M_{3X+5}(t)$ is well defined and finite when $t \in (-\frac{1}{3}, \frac{1}{3})$

(b) (7%) Find the mgf of $X + Y$.

$M_{X+Y}(t) = M_X(t)\, M_Y(t)$ if X and Y are independent

so $M_{X+Y}(t) = \frac{1}{(1+t)(1-t)} * \frac{1}{(1+t)(1-t)} = \frac{1}{(1+t)^2(1-t)^2}$

range of t: $\{-1 < t < 1\} \cap \{-1 < t < 1\} = \{-1 < t < 1\}$

so $M_{X+Y}(t)$ is well-defined and finite when $t \in (-1, 1)$

-0

Problem 5. (16%) Suppose X is a random variable with moment generating function (mgf) given by

$$M(t) = e^{\beta(e^{\lambda t}-1)} = \exp[\beta(e^{\lambda t}-1)] \qquad \text{where } \beta > 0.$$

Use the mgf to find the mean and variance of X.

$$M(t) = e^{\beta(e^{\lambda t}-1)} = \exp[\beta(e^{\lambda t}-1)] \qquad \beta > 0$$

$$EX = M'(t)\big|_{t=0} = \exp[\beta(e^{\lambda t}-1)]\,\beta\lambda e^{\lambda t}\big|_{t=0}$$

$$= \lambda\beta$$

$$EX^2 = M''(t)\big|_{t=0} = \frac{d\,[e^{\beta(e^{\lambda t}-1)}\lambda\beta e^{\lambda t}]}{dt}\Big|_{t=0}$$

$$= \beta\lambda\,[e^{\beta(e^{\lambda t}-1)}\beta\lambda e^{\lambda t} e^{\lambda t} + e^{\beta(e^{\lambda t}-1)}\lambda e^{\lambda t}]\big|_{t=0}$$

$$= \beta\lambda\,[\beta\lambda + \lambda] = \beta\lambda^2(\beta+1)$$

$$var(X) = EX^2 - (EX)^2 = \beta\lambda^2(\beta+1) - (\lambda\beta)^2 = \beta\lambda^2$$

so $EX = \lambda\beta$ is the mean. $var(X) = \lambda^2\beta$ is the variance of X.

mean= $\lambda\beta$ variance= $\lambda^2\beta$

-0

Problem 6. Let $X \sim$ Negative Binomial(r, p) according to the textbook definition (in which the possible values are $0, 1, 2, \ldots$). See the appendix for the pmf, mean, variance, and mgf.

In the following parts, compute a **numerical** answer using an appropriate **approximation**. Briefly justify the approximation that you use.

In each part, let $\mu = EX$ and $\sigma^2 = \text{Var}(X)$.

(a) (9%) Suppose $r = \mathbf{10^{10}}$ and $p = \mathbf{.25}$. Find $P\left(\mu - \frac{\sigma}{2} < X < \mu + \frac{\sigma}{2}\right)$.

$r = 10^{10}$, $p = .25$ $\quad$ r is large and p is small.

$X \sim AN(\mu, \sigma^2)$. $\quad$ $\mu = EX = \frac{r(1-p)}{p} = \frac{10^{10} * .75}{.25} = 3*10^{10}$

$\sigma^2 = var(X) = \frac{r(1-p)}{p^2} = \frac{10^{10} * .75}{.25 * .25} = 1.2 \times 10^{11}$

So $X \sim AN(3 \times 10^{10}, 1.2 \times 10^{11})$

$P(\mu - \frac{\sigma}{2} < X < \mu + \frac{\sigma}{2}) = P\left(\frac{\mu - \sigma/2 - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{\mu + \sigma/2 - \mu}{\sigma}\right)$

$\approx P(-\frac{1}{2} < Z < \frac{1}{2})$ $\quad$ $Z \sim N(0,1)$

$= \Phi(.5) - \Phi(-.5) = \Phi(.5) - [1 - \Phi(.5)]$

$= 2\Phi(.5) - 1$

$= 2 * [1 - .3085] - 1$

$= 2 * 0.6915 - 1$

$= 0.383$

$X \sim NegBin(r, p) \sim \sum_{i=1}^{r}(Z_i - 1)$, $\quad$ $Z_i \sim$ Geometric (p) $\quad EZ_i = \frac{1}{p}$

$var Z_i = \frac{1-p}{p^2}$

So, using CLT when r is large. $\quad \frac{X - (rEZ_i - r)}{r var(Z_i)} \sim AN(0, 1)$,

Here, $rEZ_i - r = \mu(X) = EX$, $\quad r var(Z_i) = var(X)$

0

[**Problem 6 continued**]

(b) (6%) Suppose $r = 10^{10}$ and $p = 1 - 10^{-10}$. Find $P\left(\mu - \frac{\sigma}{2} < X < \mu + \frac{\sigma}{2}\right)$.

$q \equiv 1 - p = 10^{-10}$, $r \cdot q = 1$, $X \sim \binom{r+x-1}{x} \cdot p^r q^x$ can be approximated by Poisson($\lambda = rq = 1$), proved as below:

$$P(X=x) = \frac{\overbrace{(x+r-1)(x-1+r-1)(x-2+r-1)\cdots(r-2)\cdot(r-1)!}}{\underbrace{r \cdot r \cdot r \cdots r}_{x \text{ terms: from } x \text{ to } 1}\,(r-1)!\,x!}(1-q)^{\frac{1}{q}\cdot rq} \cdot (rq)^x$$

as r very large, and q very small $\approx 1^x \cdot e^{-rq} \cdot \frac{(rq)^x}{x!}$ $\cancel{q e}$

Thus $\mu = \lambda = rq = 1$, $\sigma = \sqrt{\lambda} = \sqrt{rq} = 1$ (same results from table NegBinomial $\mu = \frac{r(1-p)}{p}$, $\sigma^2 = \frac{r(1-p)}{p^2}$)

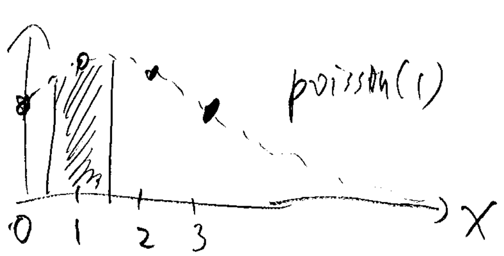
$$P\left(\mu - \frac{\sigma}{2} < X < \mu + \frac{\sigma}{2}\right) \approx P\left(\frac{1}{2} < X < \frac{3}{2}\right)$$

$$\approx 1 \cdot \text{Poission}(\lambda = 1, x = 1) = e^{-1} \cdot \frac{1^1}{1!} = e^{-1} \approx 0.36788$$

width & height of the bar

calculator

#



(CLT ~~gives a similar value, but is less accurate~~ ∵ r & p two extreme)

The remaining problems are multiple choice. Circle the single correct response. No work is required.

Problem 7. (3%) In the neighborhood of $t = 0$, which one of the following is the moment generating function (mgf) of some distribution?

a) $\dfrac{t}{1-2t}$ b) $\dfrac{2t}{1+2t}$ c) $\dfrac{2}{1-2t}$ d) $\dfrac{1}{t(1-2t)}$ e) $\dfrac{2}{t(1-t)}$ f) $\dfrac{1}{1-2t}$ g) $\dfrac{2t}{1-t}$

Problem 8. (3%) Suppose the possible values of a random variable X are $0, 1, 2, \ldots, 9, 10$. finite
For this random variable, what is the set of values of t for which the moment generating function (mgf) $M(t)$ is well-defined and finite?

a) $[0, \infty)$ b) $(-\infty, 0]$ c) $(0, \infty)$ d) $(-\infty, 0)$ e) $(-\infty, \infty)$ f) $[0, 10]$ g) $(0, 10)$

h) $[10, \infty)$ i) $(10, \infty)$ j) not enough information is given to determine the answer

Problem 9. (3%) Suppose X is a random variable, and y and z are positive values. Then

$$P(X > y \mid X > y + z)$$

$y+z > y$

is equal to which of the following?

a) 0 b) 1 c) 1/2 d) $\dfrac{P(y < X < y+z)}{P(X > y+z)}$ e) $\dfrac{P(y < X < y+z)}{P(X > y)}$

f) $\dfrac{P(X > y)}{P(y < X < y+z)}$ g) $\dfrac{P(X > y+z)}{P(y < X < y+z)}$ h) $\dfrac{P(X > y+z)}{P(X > y)}$ i) $\dfrac{P(X > y)}{P(X > y+z)}$

0