TEST #3
STA 5326 **Name:**______________________________________
December 3, 2009

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- If there is any chance for confusion, **circle your answer**. **Cross out any work you want the grader to ignore.** (The grader will deduct points if it is not clear what your answer is, or if there is erroneous work left on your paper.)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely **unless numerical answers are requested**. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **10** pages.
- There are a total of **100** points.

Problem 1. Suppose $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Define $\{X(t) : t \geq 0\}$ by

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

where $Y_1, Y_2, Y_3, \ldots$ are iid with cdf F and independent of $N(\cdot)$.

[Let $Y \sim F$. In your answers to the following, you may use $\mu \equiv EY$, $\sigma^2 \equiv \text{Var}(Y)$ or moments such EY^2, EY^3, etc.]

(a) (9%) Suppose $0 < s < u$. Determine $\text{Cov}(X(s), X(u))$.

[**Problem 1 continued**]

(b) (9%) Compute $\mathrm{Cov}(N(t), X(t))$.

Problem 2. Suppose (X, Y) has joint density

$$f_{X,Y}(x, y) = \frac{1}{2}(x + y)e^{-(x+y)} \quad \text{for } 0 < x < \infty\,,\ 0 < y < \infty.$$

Define $U = X + Y$ and $V = X/(X + Y)$.

(a) (12%) Find the joint density of (U, V). [Your answer should include the support.]

[**Problem 2 continued**]

(b) (4%) Find the density of U. [Your answer should include the support.]

(c) (4%) Find the density of V. [Your answer should include the support.]

Problem 3. Suppose X has a Beta$(1, 2)$ distribution with density $f_X(x) = 2(1 - x)$ for $0 < x < 1$, and given X, the random variable Y has a binomial distribution with n trials and success probability equal to X.

(a) (8%) Find EY.

(b) (9%) Find $\mathrm{Var}(Y)$.

Problem 4. (9%) A random point (X, Y) is distributed uniformly on the **triangle** with vertices $(2, 0)$, $(0, 2)$, and $(-2, 0)$. Find the probability that $X^2 + Y^2 < 1$.

Problem 5. Let X_1, X_2, and X_3 be uncorrelated random variables, each with mean μ and variance σ^2. Define $W = X_1 + X_2 + 8$ and $Y = X_2 + X_3 + 9$.

(a) (8%) Find $\mathrm{Cov}(W, Y)$.

(b) (5%) Find $\rho = \mathrm{Corr}(W, Y)$, the correlation between W and Y.

For the problems on this page, you should show a little work, but not too much.

Problem 6. (6%) Vehicles pass a certain corner according to a Poisson process with rate 60 vehicles per hour. Each vehicle that passes is either a car or bicycle with probabilities 5/6 and 1/6, respectively, independently of everything that has previously occurred. Let $B(t)$ be the number of bicycles that have passed by time t (in hours). What is the distribution of $B(2)$? (Specify the name of the distribution and the values of any parameters.)

The distribution of $\boldsymbol{B(2)}$ is ________________________________

Problem 7. (6%) Patients arrive at an emergency room according to a Poisson process with rate 1.752 patients per hour. Given that **exactly 3** patients arrive in the next hour, what is the probability that **none** of them arrive in the last 10 minutes of the hour?

probability = ________________________

Problem 8. The transition probability matrix P for a Markov chain $X_0, X_1, X_2, \ldots$ with state space $\{1, 2, 3, 4\}$ is given below along with the matrix products P^2 and P^3.

$$P = \begin{pmatrix} 0.11 & 0.36 & 0.23 & 0.3 \\ 0.01 & 0.03 & 0.55 & 0.41 \\ 0.07 & 0.75 & 0.14 & 0.04 \\ 0.28 & 0.19 & 0.48 & 0.05 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.1158 & 0.2799 & 0.3995 & 0.2048 \\ 0.1547 & 0.4949 & 0.2926 & 0.0578 \\ 0.0362 & 0.1603 & 0.4674 & 0.3361 \\ 0.0803 & 0.476 & 0.2601 & 0.1836 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.1008 & 0.3886 & 0.3348 & 0.1757 \\ 0.0586 & 0.301 & 0.3765 & 0.2639 \\ 0.1324 & 0.4322 & 0.3233 & 0.1121 \\ 0.0832 & 0.2731 & 0.4048 & 0.2388 \end{pmatrix}$$

Suppose the chain starts from state 1.

(a) (4%) What is the probability the chain goes from state 1 to 2 to 3 to 4? In terms of the two different notations mentioned in lecture, I am asking you to find the value of

$$P_1(X_1 = 2, X_2 = 3, X_3 = 4) = P(X_1 = 2, X_2 = 3, X_3 = 4 \,|\, X_0 = 1)\,.$$

(Show a little work.)

probability=____________________

(b) (3%) What is the probability the chain is in state 4 at time 3? In terms of the two different notations mentioned in lecture, I am asking you to find the value of

$$P_1(X_3 = 4) = P(X_3 = 4 \,|\, X_0 = 1)\,.$$

(No work is required.)

probability=____________________

Problem 9. (4%) Suppose X and Y are iid with the common density $f(x) = 6x(1-x)$ for $0 < x < 1$. The density of $Z = X + Y$ may be obtained by computing the convolution integral

$$f_Z(z) = \int_a^b 6x(1-x)\,6(z-x)(1-z+x)\,dx\,.$$

For $\mathbf{1 < z < 2}$, what are the limits of integration a and b? (No work is required.)

$\boldsymbol{a} =$________________ $\boldsymbol{b} =$________________