TEST #3
STA 5326 **Name:**____________________________________
December 3, 2009

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- If there is any chance for confusion, **circle your answer**. **Cross out any work you want the grader to ignore.** (The grader will deduct points if it is not clear what your answer is, or if there is erroneous work left on your paper.)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely **unless numerical answers are requested**. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **10** pages.
- There are a total of **100** points.

Problem 1. Suppose $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Define $\{X(t) : t \geq 0\}$ by

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

where $Y_1, Y_2, Y_3, \ldots$ are iid with cdf F and independent of $N(\cdot)$.

[Let $Y \sim F$. In your answers to the following, you may use $\mu \equiv EY$, $\sigma^2 \equiv \text{Var}(Y)$ or moments such EY^2, EY^3, etc.]

(a) (9%) Suppose $0 < s < u$. Determine $\text{Cov}(X(s), X(u))$.

$$\begin{aligned} cov(X(s), X(u)) &= cov(X(s), X(u) - X(s) + X(s)) \\ &= cov(X(s), X(u) - X(s)) + cov(X(s), X(s)) \\ &= 0 + var(X(s)). \end{aligned}$$

< as $X(s)$ is independent of $X(u) - X(s)$ > , $0 < s < u$

$$\begin{aligned} var(X(s)) &= var\left(\sum_{i=1}^{N(s)} Y_i\right) \\ &= var\, E(X(s) \mid N(s)) + E\, var(X(s) \mid N(s)). \end{aligned}$$

$E(X(s) \mid N(s) = n) = E\left(\sum_{i=1}^{n} Y_i \mid N(s) = n\right) = nEY_i = n\mu \Rightarrow E(X(s) \mid N(s)) = N(s)\mu$

$Var(X(s) \mid N(s) = n) = var\left(\sum_{i=1}^{n} Y_i \mid N(s) = n\right) = n\, var(Y_i) = n\sigma^2 \Rightarrow var(X(s) \mid N(s)) = N(s)\sigma^2$

$$\begin{aligned} \Rightarrow var(X(s)) &= var(N(s)\mu) + E(N(s)\sigma^2) \\ &= \mu^2 var(N(s)) + \sigma^2 E(N(s)). \qquad (N(\cdot) \sim PP(\lambda)) \\ &= \mu^2(\lambda s) + \sigma^2 \lambda s. \\ &= \lambda s(\mu^2 + \sigma^2) = \lambda s EY^2, \qquad \mu = EY, \ \sigma^2 = var(Y). \end{aligned}$$

So $cov(X(s), X(u)) = var(X(s)) = \lambda s EY^2, \quad Y \sim F.$

9

[**Problem 1 continued**]

(b) (9%) Compute $\mathrm{Cov}(N(t), X(t))$.

$$Cov(N(t), X(t)) = E[(N(t)-EN(t))(X(t)-EX(t))]$$

$$= EN(t)X(t) - EN(t)EX(t)$$

$$EN(t)X(t) = E\,E(N(t)X(t)\,|\,N(t))$$

$$E(N(t)X(t)\,|\,N(t)=n) = E(n\sum_{i=1}^{n} Y_i) = n^2 EY_i$$

$$\Rightarrow E(N(t)X(t)\,|\,N(t)) = N^2(t)EY_i$$

Nevermind. You are correct.

$$\Rightarrow EN(t)X(t) = E(N^2(t)EY_i) = EY_i\,EN^2(t)$$

$$EX(t) = E[E(X(t)\,|\,N(t))]$$

$$E(X(t)\,|\,N(t)=n) = E(\sum_{i=1}^{n} Y_i\,|\,N(t)=n) = nEY_i$$

$$\Rightarrow E(X(t)\,|\,N(t)) = N(t)EY_i$$

$$\Rightarrow EX(t) = E(N(t)EY_i) = EY_i \cdot EN(t) \qquad Y_i \sim F$$

So, $Cov(N(t), X(t)) = EY_i\,EN^2(t) - EN(t)\cdot EY_i \cdot EN(t)$

$$= EY_i\,(EN^2(t) - (EN(t))^2)$$

$$= EY_i\,var(N(t))$$

$$= \lambda t \cdot EY_i = \lambda t\,EY \qquad Y \sim F.$$

Problem 2. Suppose (X, Y) has joint density

$$f_{X,Y}(x,y) = \frac{1}{2}(x+y)e^{-(x+y)} \quad \text{for } 0 < x < \infty,\ 0 < y < \infty.$$

Define $U = X + Y$ and $V = X/(X+Y)$.

(a) (12%) Find the joint density of (U, V). [Your answer should include the support.]

$$\left.\begin{aligned} U &= X+Y \in (0, +\infty) \\ V &= X/(X+Y) \in (0, 1) \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} X &= UV \\ Y &= U - UV \end{aligned}\right.$$

$$|J| = \left|\frac{\partial(X,Y)}{\partial(U,V)}\right| = \left|\begin{matrix} V & U \\ 1-V & -U \end{matrix}\right| = |-UV - U(1-V)| = U$$

$$f_{U,V}(u,v) = f_{X,Y} \cdot |J| = \frac{1}{2}(uv + u - uv)e^{-u}\, u$$

$$= \frac{1}{2}ue^{-u} \cdot u = \frac{1}{2}u^2e^{-u} \quad \text{for } \begin{aligned} 0<u<\infty \\ 0<v<1 \end{aligned}$$

so $f_{U,V}(u,v) = \frac{1}{2}u^2e^{-u} I_{(0,\infty)}(u) \cdot 1 \cdot I_{(0,1)}(v)$.

[**Problem 2 continued**]

(b) (4%) Find the density of U. [Your answer should include the support.]

$$f_U(u) = \int f_{U,V}(u,v)\,dv$$

$$= \int_0^1 \tfrac{1}{2}u^2 e^{-u} I_{(0,+\infty)}(u)\,dv$$

$$= \tfrac{1}{2}u^2 e^{-u} I_{(0,+\infty)}(u).$$

(c) (4%) Find the density of V. [Your answer should include the support.]

$$f_V(v) = \int f_{U,V}(u,v)\,du$$

$$= \int_0^{+\infty} \tfrac{1}{2}u^2 e^{-u} I_{(0,1)}(v)\,du.$$

$$= I_{(0,1)}(v)\left[-e^{-u}\cdot\tfrac{1}{2}u^2\Big|_0^{+\infty} + \int_0^{\infty} e^{-u}\cdot u\,du\right]$$

$$= I_{(0,1)}(v)\left[\int_0^{\infty} u e^{-u}\,du\right] = I_{(0,1)}(v)\left[-e^{-u}u\Big|_0^{+\infty} + \int_0^{+\infty} e^{-u}\,du\right]$$

$$= I_{(0,1)}(v)\int_0^{+\infty} e^{-u}\,du = I_{(0,1)}(v)\left(-e^{-u}\Big|_0^{+\infty}\right)$$

$$= I_{(0,1)}(v)$$

so V has the uniform distribution on $(0,1)$

-0

Problem 3. Suppose X has a Beta$(1,2)$ distribution with density $f_X(x) = 2(1-x)$ for $0 < x < 1$, and given X, the random variable Y has a binomial distribution with n trials and success probability equal to X.

(a) (8%) Find EY.

$Y|X \sim$ Binomial (n, X) ; $X \sim$ Beta $(1,2)$, $\alpha=1$, $\beta=2$

$EY = E(EY|X)$; $E(Y|X) = nX$, n is known.

$\Rightarrow EY = E(nX) = nEX = n \cdot \frac{\alpha}{\alpha+\beta} = n \cdot \frac{1}{1+2} = \frac{n}{3}$

(b) (9%) Find Var(Y).

$var(Y) = E\, var(Y|X) + var(EY|X)$.

$var(Y|X) = nX(1-X)$, $EY|X = nX$

$\Rightarrow var(Y) = EnX(1-X) + var(nX)$

$= nEX - nEX^2 + n^2 var(X)$.

$EX^2 = var(X) + (EX)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \left(\frac{\alpha}{\alpha+\beta}\right)^2$

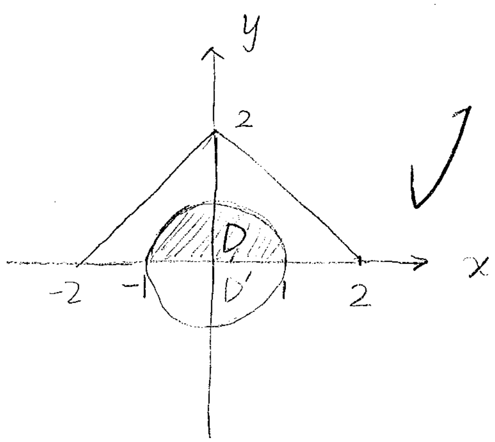
$= \frac{1*2}{(1+2)^2(1+2+1)} + \left(\frac{1}{1+2}\right)^2 = \frac{2}{36} + \frac{1}{9} = \frac{1}{6}$

$var(X) = \frac{1}{18}$

$\Rightarrow var(Y) = n \cdot \frac{1}{3} - n \cdot \frac{1}{6} + n^2 \cdot \frac{1}{18} = \frac{1}{6}n + \frac{1}{18}n^2$

Problem 4. (9%) A random point (X, Y) is distributed uniformly on the **triangle** with vertices $(2, 0)$, $(0, 2)$, and $(-2, 0)$. Find the probability that $X^2 + Y^2 < 1$.

$f_{X,Y}(x,y) = \frac{1}{4}$ for x, y on the triange with vertices $(2,0)$ $(0,2)$ & $(-2,0)$



$P(X^2+Y^2<1) = \iint_{D'\cup D} f_{X,Y}(x,y)\,dxdy$. D is the shadow part

we just need to get the area of D, <as (X,Y) is distributed uniformly on the triangle>, $f_{X,Y}(x,y)=0$ for (X,Y) on D'

so $P(X^2+Y^2<1) = \iint_D f_{X,Y}(x,y)\,dxdy$.

$$= \frac{\text{Area}(D)}{\text{Area}(\text{Triangle})} = \frac{\pi \cdot 1^2/2}{\frac{1}{2} * 2 * (2-(-2))}$$

$$= \frac{\pi/2}{\frac{1}{2} * 2 * 4} = \frac{\pi}{8}$$

0

Problem 5. Let X_1, X_2, and X_3 be uncorrelated random variables, each with mean μ and variance σ^2. Define $W = X_1 + X_2 + 8$ and $Y = X_2 + X_3 + 9$.

(a) (8%) Find $\mathrm{Cov}(W, Y)$.

$$\mathrm{cov}(W, Y) = \mathrm{cov}(X_1 + X_2 + 8,\ X_2 + X_3 + 9)$$
$$= \mathrm{cov}(X_1 + X_2,\ X_2 + X_3)$$
$$= \mathrm{cov}(X_1, X_2) + \mathrm{cov}(X_2, X_2) + \mathrm{cov}(X_1, X_3) + \mathrm{cov}(X_2, X_3).$$

As X_1, X_2, X_3 be uncorrelated random variables,

$$\mathrm{cov}(X_1, X_2) = \mathrm{cov}(X_1, X_3) = \mathrm{cov}(X_2, X_3) = 0$$

$$\mathrm{cov}(W, Y) = \mathrm{cov}(X_2, X_2) = \mathrm{var}(X_2) = \sigma^2$$

(b) (5%) Find $\rho = \mathrm{Corr}(W, Y)$, the correlation between W and Y.

$$\rho = \mathrm{corr}(W, Y) = \frac{\mathrm{cov}(W, Y)}{\sqrt{\mathrm{var}(W)\,\mathrm{var}(Y)}}$$

$$\mathrm{var}(W) = \mathrm{var}(X_1 + X_2 + 8) = \mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + 2\,\mathrm{cov}(X_1, X_2) + \mathrm{var}(X_2)$$
$$= \mathrm{var}(X_1) + \mathrm{var}(X_2) = \sigma^2 + \sigma^2 = 2\sigma^2.$$
$$\mathrm{var}(Y) = \mathrm{var}(X_2 + X_3 + 9) = \mathrm{var}(X_2 + X_3) = \mathrm{var}(X_2) + 2\,\mathrm{cov}(X_2, X_3) + \mathrm{var}(X_3)$$
$$= \mathrm{var}(X_2) + \mathrm{var}(X_3) = \sigma^2 + \sigma^2 = 2\sigma^2$$

$$\rho = \frac{\mathrm{cov}(W, Y)}{\sqrt{\mathrm{var}(W)\,\mathrm{var}(Y)}} = \frac{\sigma^2}{\sqrt{2\sigma^2}\sqrt{2\sigma^2}} = \frac{\sigma^2}{2\sigma^2} = \frac{1}{2}$$

−0

For the problems on this page, you should show a little work, but not too much.

Problem 6. (6%) Vehicles pass a certain corner according to a Poisson process with rate 60 vehicles per hour. Each vehicle that passes is either a car or bicycle with probabilities 5/6 and 1/6, respectively, independently of everything that has previously occurred. Let $B(t)$ be the number of bicycles that have passed by time t (in hours). What is the distribution of $B(2)$? (Specify the name of the distribution and the values of any parameters.)

$B(t)$ is the # of bicycles passed by time t

$B(\cdot) \sim PP(\lambda \cdot \frac{1}{6}) = PP(10)$.

$B(t) \sim \text{poisson}(\lambda_1 t)$, $\lambda_1 = 10$

so $B(2) \sim \text{poisson}(20)$, poisson distribution with rate 20

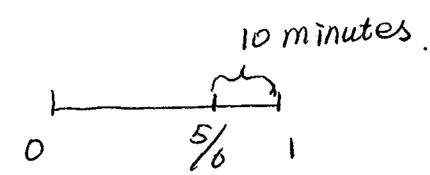
The distribution of $\boldsymbol{B(2)}$ is poisson (20)

Problem 7. (6%) Patients arrive at an emergency room according to a Poisson process with rate 1.752 patients per hour. Given that **exactly 3** patients arrive in the next hour, what is the probability that **none** of them arrive in the last 10 minutes of the hour?

$P(N(1) - N(\frac{5}{6}) = 0 \mid N(1) = 3)$

$= P(N(\frac{5}{6}) = 3 \mid N(1) = 3)$

$= (\frac{5}{6})^3$



probability = $(\frac{5}{6})^3$

-0

Problem 8. The transition probability matrix P for a Markov chain $X_0, X_1, X_2, \ldots$ with state space $\{1, 2, 3, 4\}$ is given below along with the matrix products P^2 and P^3.

$$P = \begin{pmatrix} 0.11 & 0.36 & 0.23 & 0.3 \\ 0.01 & 0.03 & 0.55 & 0.41 \\ 0.07 & 0.75 & 0.14 & 0.04 \\ 0.28 & 0.19 & 0.48 & 0.05 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.1158 & 0.2799 & 0.3995 & 0.2048 \\ 0.1547 & 0.4949 & 0.2926 & 0.0578 \\ 0.0362 & 0.1603 & 0.4674 & 0.3361 \\ 0.0803 & 0.476 & 0.2601 & 0.1836 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.1008 & 0.3886 & 0.3348 & 0.1757 \\ 0.0586 & 0.301 & 0.3765 & 0.2639 \\ 0.1324 & 0.4322 & 0.3233 & 0.1121 \\ 0.0832 & 0.2731 & 0.4048 & 0.2388 \end{pmatrix}$$

Suppose the chain starts from state 1.

(a) (4%) What is the probability the chain goes from state 1 to 2 to 3 to 4? In terms of the two different notations mentioned in lecture, I am asking you to find the value of

$$P_1(X_1 = 2, X_2 = 3, X_3 = 4) = P(X_1 = 2, X_2 = 3, X_3 = 4 \mid X_0 = 1).$$

(Show a little work.) $= P_{12} P_{23} P_{34}$

$= P(X_1=2 \mid X_0=1)\, P(X_2=3 \mid X_1=2, X_0=1)\, P(X_3=4 \mid X_2=3, X_1=2, X_0=1)$

$= P(X_1=2 \mid X_0=1)\, P(X_2=3 \mid X_1=2)\, P(X_3=4 \mid X_2=3)$

$= P_{12} P_{23} P_{34}$

probability= $.36 * .55 * .04 = .00792$ ✓

(b) (3%) What is the probability the chain is in state 4 at time 3? In terms of the two different notations mentioned in lecture, I am asking you to find the value of

$$P_1(X_3 = 4) = P(X_3 = 4 \mid X_0 = 1). = P_{14}^{(3)} = (P^3)_{14}$$

(No work is required.)

probability= $.1757$ ✓

Problem 9. (4%) Suppose X and Y are iid with the common density $f(x) = 6x(1-x)$ for $0 < x < 1$. The density of $Z = X + Y$ may be obtained by computing the convolution integral

$$f_Z(z) = \int_a^b 6x(1-x)\,6(z-x)(1-z+x)\,dx.$$

For $1 < z < 2$, what are the limits of integration a and b? (No work is required.)

$0 < x < 1$, $0 < z - x < 1$ $\Rightarrow$ $z-1 < x < z$ $a = \max(0, z-1)$, $b = \min(1, z)$

for $1 < z < 2$, $a = \max(0, z-1) = z-1$, $b = \min(1, z) = 1$

$0 < z-1 < 1$

$a =$ $z-1$ ✓ $b =$ 1 ✓

Yes!

0