TEST #1
STA 5326 **Name:**____________________________________
September 27, 2010

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.
- There are a total of **100** points.

Problem 1. In a simple game, 50 balls labeled

$A0$	$A1$	$A2$	$A3$	$A4$	$A5$	$A6$	$A7$	$A8$	$A9$
$B0$	$B1$	$B2$	$B3$	$B4$	$B5$	$B6$	$B7$	$B8$	$B9$
$C0$	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$	$C7$	$C8$	$C9$
$D0$	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$	$D7$	$D8$	$D9$
$E0$	$E1$	$E2$	$E3$	$E4$	$E5$	$E6$	$E7$	$E8$	$E9$

are placed in a jar, and a player draws out **three** balls at random with**OUT** replacement. A player wins cash prizes if any of the following occur:

- All three letters are the same.
- There are repeated digits. (Two or more of the digits take the same value.)
- The digits can be placed in a sequence.
  (There are 8 possible sequences: 012, 123, ..., 678, 789.)

For a person playing this game, find the probability of each of the following events.

(a) (6%) All three letters are the same.

let A = event you select 3 A's
B = event 3 B's, etc

$$\frac{\binom{5}{1}\binom{10}{3}}{\binom{50}{3}} \checkmark$$

$\#(A \cup B \cup C \cup D \cup E) = \#(A) + \#(B) + \dots$

$\#(A) = \binom{10}{3}$ for 5 letters!

$\binom{50}{3}$ = # ways to draw 3 balls from 50 w/out replace $\checkmark$

$$= \frac{5\binom{10}{3}}{\binom{50}{3}} \approx 0.0306 \checkmark$$

(b) (6%) There are repeated digits.

P(repeated digits) = 1 − P(No repeated digits) $\checkmark$

$$1 - \frac{\binom{10}{3}\binom{5}{1}^3}{\binom{50}{3}} \checkmark \approx 0.239$$

(12)

[Problem 1 continued]

(c) (4%) The digits can be placed in a sequence.

012, 123, ... $\Rightarrow$ there are 8 possible sequences

consider a particular sequence, say 012:

$P(012) = \dfrac{\binom{5}{1}\binom{5}{1}\binom{5}{1}}{\binom{50}{3}}$ $\longrightarrow$ since there are $\binom{5}{1}$ ways to get each digit ✓

Now each sequence is equally likely yielding

$P(\text{sequence}) = \dfrac{8\binom{5}{1}^3}{\binom{50}{3}} = \dfrac{8(125)}{\binom{50}{3}} \approx 0.05$ ✓

(d) (8%) The player wins nothing at all (i.e., none of the three winning situations occur).

Let The 3 winning events be denoted

A = letters are the same $\rightarrow P(A) \approx 0.0306$

B = Repeated digits $\rightarrow P(B) \approx .234$

C = Sequence $\rightarrow P(C) \approx 0.05$

$P(\text{win nothing}) = 1 - P(\underbrace{\text{win something}}_{\text{win at least one thing}})$

b/c there are 5 letters & 8 sequences each:

$P(A \cap C) = \dfrac{8(5)}{\binom{50}{3}} \approx .002$

$= 1 - \left[P(A \cup B \cup C)\right]$

$*\ P(A \cap C) = \dfrac{40}{\binom{50}{3}}$

✓ $= 1 - \left[P(A) + P(B) + P(C) - \cancel{P(A \cap B)} - P(A \cap C) - \cancel{P(B \cap C)} + \ldots \right.$
$\left. \ldots + \cancel{P(A \cap B \cap C)}\right]$

NOTE: A & B ARE DISJOINT THUS $P(A \cap B) = \phi$ & $P(A \cap B \cap C) = \phi$

ALSO B & C ARE DISJOINT THUS $P(B \cap C) = \phi$ ✓

$P(\text{win nothing}) = 1 - [P(A) + P(B) + P(C) - P(A \cap C)] \approx 1 - [.0306 + .234 + 0.05 - .002$

(12)

Problem 2. Suppose X has pdf (density) defined by

$$f(x) = \begin{cases} \frac{c}{x} & \text{for } 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) (6%) What is the value of the constant c?

$$1 = \int_{-\infty}^{\infty} f(x) = \int_1^2 \frac{c}{x}dx = c\ln x \Big|_1^2 = c\ln 2 - \cancel{c\ln 1} = c\ln 2$$

$$c\ln 2 = 1 \Rightarrow c = \frac{1}{\ln 2}$$

If you could not find the value of c, then, in the remaining parts, just leave it as c in your answers.

(b) (6%) Calculate $P(1.25 < X < 1.75)$.

$$\int_{1.25}^{1.75} \frac{c}{x}dx = c\ln x \Big|_{1.25}^{1.75} = c\left[\ln 1.75 - \ln 1.25\right]$$

$$= \frac{1}{\ln 2}(\ln 1.75 - \ln 1.25) \approx 0.485$$

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[Problem 2 continued]

(c) (6%) Calculate $E(\log X)$.

$$= E(\ln X) = \int_1^2 \ln X \cdot \frac{c}{X} dx = c\int_1^2 \frac{\ln x}{x} dx$$

let $U = \ln x \quad du = \frac{1}{x}$

$dv = \frac{1}{x} \quad v = \ln x$

$$c\left[\int_1^2 \frac{\ln x}{x} dx\right] = c\left[\left(\ln x \Big|_1^2\right)^2 - \int_1^2 \frac{\ln x}{x} dx\right] \Rightarrow 2\int_1^2 \frac{\ln x}{x} dx = \left(\ln x \Big|_1^2\right)^2$$

Answer

$$\Rightarrow \int_1^2 \frac{\ln x}{x} dx = \frac{(\ln 2 - \cancel{\ln 1})^2}{2} = \frac{(\ln 2)^2}{2} \quad \text{thus } E(\ln X) = \frac{c(\ln 2)^2}{2} = \frac{(\ln 2)^{\cancel{2}}}{2\cancel{\ln 2}} \frac{\ln 2}{2}$$

(d) (6%) Calculate $\mathrm{Var}(X^2)$.

$$\mathrm{Var}(x^2) = E[(x^2)^2] - [E(x^2)]^2$$

$$= E(x^4) - (E(x^2))^2$$

$$E(x^4) = \int_1^2 x^4 \cdot \frac{c}{x} dx = c\int_1^2 x^3 dx = c\left(\frac{x^4}{4}\Big|_1^2\right) = c\left[\frac{16-1}{4}\right] = \frac{1}{\ln 2}\left(\frac{15}{4}\right)$$

$$E(x^2) = \int_1^2 x^2 \cdot \frac{c}{x} dx = c\int_1^2 x\, dx = c\left(\frac{x^2}{2}\Big|_1^2\right) = c\left(\frac{4-1}{2}\right) = \frac{1}{\ln 2}\left(\frac{3}{2}\right)$$

$$\mathrm{Var}(x^2) = \frac{1}{\ln 2}\left(\frac{15}{4} - \frac{3}{2}\right) = \frac{1}{\ln 2}\left(\frac{15-6}{4}\right) = \frac{9}{4\ln 2}$$

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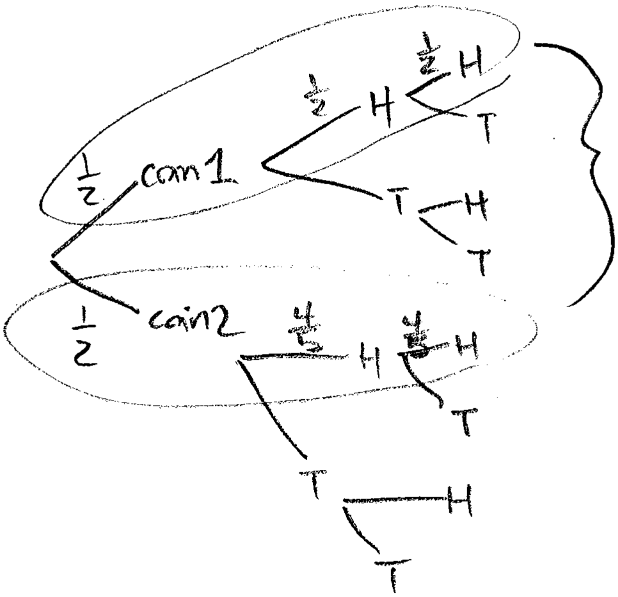
Problem 3. (12%) A hat contains two coins. One is a fair coin with $P(\text{head}) = 1/2$, but the other is biased with $P(\text{head}) = 4/5$. A coin is chosen at random from the hat and tossed two times. If both tosses are heads, what is the probability you have drawn the biased coin?

For coin #1 $P(H_1) = \frac{1}{2}$

coin #2 $P(H_2) = \frac{4}{5}$

$$P(\underbrace{\text{Drawn biased coin}}_{\text{denote event } B} \mid \underbrace{\text{first two tosses turn up Heads}}_{\text{denote event } HH}) = \frac{P(B \cap HH)}{P(HH)}$$



$$P(HH) = \left(\tfrac{1}{2}\right)^3 + \tfrac{1}{2}\left(\tfrac{4}{5}\right)^2$$

prob. choosing coin 2 & flipping H's twice

The prob. of choosing coin 1 & flipping H's twice

$$\text{Thus } \frac{P(B \cap HH)}{P(HH)} = \frac{\frac{1}{2}\left(\frac{4}{5}\right)^2}{\left[\left(\frac{1}{2}\right)^3 + \frac{1}{2}\left(\frac{4}{5}\right)^2\right]} = \frac{\left(\frac{16}{50}\right)}{\left(\frac{1}{8} + \frac{16}{50}\right)}$$

$$\frac{\left(\frac{64}{200}\right)}{\left(\frac{25+64}{200}\right)} = \frac{64}{89} \simeq .719$$

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Problem 4. (8%) **Four** prisoners, A, B, C, and D, are on death row. The governor decides to pardon one of the **four** and chooses at random the prisoner to pardon. He informs the warden of his choice but requests that the name be kept secret for a few days.

The next day, A tries to get the warden to tell him who had been pardoned. The warden refuses. A then asks the warden to give him the name of one prisoner (B or C or D) who will be executed. The warden thinks for a moment and then says that B is to be executed. A knows that the warden is too lazy to bother choosing a name at random and will simply pick the first person on the list $\{B, C, D\}$ who is going to be executed. So, given that the warden says B, what is the probability that A will be pardoned?

denote A denote W

$$P(\text{A pardoned} \mid \text{warden says B}) = \frac{P(\text{A pardon} \cap \text{warden says B})}{P(\text{warden says B})}$$

$$P(A \cap W) = P(W|A)P(A) = (1)\left(\frac{1}{4}\right)$$ ← b/c warden will pick 1st name on list.

$$P(W) = P(W \cap A) + P(W \cap B) + P(W \cap C) + P(W \cap D)$$

B pardoned C pardoned D pardoned

$$= \frac{1}{4} + \phi + P(W|C)P(C) + P(W|D)P(D)$$

$$= \frac{1}{4} + \phi + (1)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{4}\right)$$

Thus, $$P(\text{A pardoned} \mid \text{Warden says B}) = \frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{1}{3}$$

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Problem 5. Let X have pdf (density) $f_X(x) = 5e^{-5x}$, $0 < x < \infty$.

(a) (8%) Find the density of $Y = 3X + 2$. (Make sure to specify where it is positive.)

Note $Y = 3x+2$ is monotonic (increasing) continuous & differentiable

for $0 < x < \infty \Rightarrow 2 < y < \infty$

$g^{-1}(y) = \frac{(y-2)}{3}$, using Formula approach we have

$$f_Y(y) = \begin{cases} \frac{5}{3} e^{-5\left(\frac{y-2}{3}\right)} & \text{on } 2 < y < \infty \\ \emptyset & \text{, else} \end{cases}$$

$*$ where $\frac{dg^{-1}(y)}{dy} = \frac{1}{3}$

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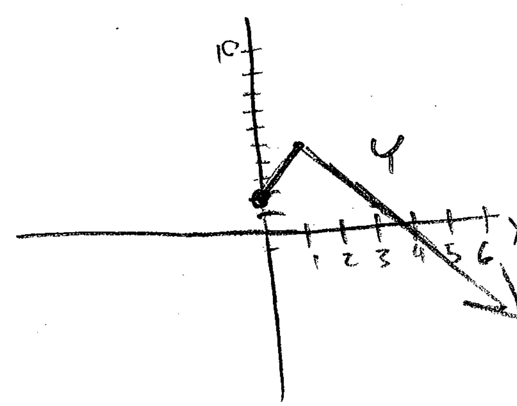
(b) (8%) Find the density of $Y = \begin{cases} 3X + 2 & \text{for } X < 1, \\ 7 - 2X & \text{for } X \geq 1. \end{cases}$ (And specify where it is positive.)

let

$A_1 = (0, 1)$ $\quad g_1^{-1}(y) = \left(\frac{y-2}{3}\right)$ $\quad \Big\}\ \frac{5}{3} e^{-5\left(\frac{y-2}{3}\right)}$, $2 < y < 5$

$B_1 = (2, 5)$ $\quad \frac{dg_1^{-1}(y)}{dy} = \frac{1}{3}$



$A_2 = (1, \infty)$ $\quad g_2^{-1}(y) = \left(\frac{7-y}{2}\right)$ $\quad \Big\}\ 5e^{-5\left(\frac{7-y}{2}\right)} \cdot \left|-\frac{1}{2}\right|$ $\quad y < 5$

$B_2 = (-\infty, 5)$ $\quad \frac{dg_2^{-1}(y)}{dy} = -\frac{1}{2}$

$$f_Y(y) = \begin{cases} \frac{5}{3} e^{-5\left(\frac{y-2}{3}\right)} + \frac{5}{2} e^{-5\left(\frac{7-y}{2}\right)}, & \text{for } 2 < y < 5 \\ \frac{5}{2} e^{-5\left(\frac{7-y}{2}\right)}, & \text{for } y < 2 \\ \emptyset & \text{, else} \end{cases}$$

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Problem 6. (12%) Suppose the random variable X has pdf (density)

$$f_X(x) = \begin{cases} 3/8 & \text{for } 0 < x \le 2, \\ 1/8 & \text{for } 2 < x \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Let F_X and P_X be the cdf and induced probability function of X, respectively, and F'_X be the derivative of the cdf.
Fill in the blanks below with the correct value or state that the required value does not exist.
(No work is required. Each blank is worth 2%.)

$F_X(3) =$ $\frac{7}{8}$ ✓

$P_X(\{2\}) =$ $\emptyset$ ✓

$P_X(\,(2,4]\,) =$ $\frac{1}{4}$ ✓

$F'_X(1) =$ $\frac{3}{8}$ ✓

$F'_X(2) =$ D.N.E. (although $\lim_{x \to 2^+} = \frac{1}{8}$)

Let $Y = F_X(X)$. Then $P(Y \le 0.75) =$ 0.75 ✓

Problem 7. (4%) A **Pólya** urn contains 5 red balls and 2 white balls. If 3 balls are drawn (in sequence) from this **Pólya** urn, what is the probability that all 3 are red? (No work is required.)

Pólya = balls is replaced AND a ball of the same color is added

$$P(3 \text{ red balls drawn}) = \frac{5}{7} \cdot \frac{6}{8} \cdot \frac{7}{9} = \frac{210}{504} \approx .42$$

✓

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