TEST #2
STA 5326 **Name:**________________________________
November 1, 2010

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** pages.
- There are a total of **100** points.

Problem 1.

(a) (4%) Give a definition for the gamma **function**, $\Gamma(\alpha)$, for $\alpha > 0$.

-2

$\cancel{\Gamma(\alpha)} = \cancel{(\alpha-1)\Gamma(\alpha-1)}$ $\int_0^\infty t^{\alpha-1} e^{-t}\,dt$

(b) (11%) Suppose X has a gamma **distribution**, $X \sim \text{Gamma}(\alpha, \beta)$. With**out** using mgf's, derive a general formula for EX^k, the k^{th} moment of X.

$X \sim \text{Gamma}(\alpha, \beta)$

$$EX^k = \int_0^\infty x^k \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\,dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha+k-1} e^{-x/\beta}\,dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha+k-1} e^{-x/\beta}\,dx$$

$$= \frac{\Gamma(\alpha+k)\,\beta^{\alpha+k}}{\Gamma(\alpha)\,\beta^\alpha}$$

$$= \beta^k \left((\alpha+k-1) \times (a+k-2) \cdots \times a \right)$$

13

Problem 2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with cumulative distribution function (cdf) $F(x) = 1 - (1-x)^k$ for $0 \le x \le 1$.

(a) (7%) Define $Y = \max X_i$. Find the **density** (**pdf**) of Y.

$P(Y < x) = P(\text{each of } X_i < x)$, since X_i are iid r.vs

$P(Y < x) = [1 - (1-x)^k]^n$

pdf: $\dfrac{d\,P(Y<x)}{dx} = n\,[1-(1-x)^k]^{n-1} \cdot k(1-x)^{k-1}$

$= kn\,(1-(1-x)^k)^{n-1}\,(1-x)^{k-1}$

for $0 \le x \le 1$

otherwise it's 0.

(b) (7%) Define $Z = \min X_i$. Find the **density** (**pdf**) of Z.

$Z = \min X_i$

$P(Z \geq x) = P(\text{each of } X_i \geq x)$

$= [1 - F(x)]^n$

$= (1-x)^{kn}$

$P(Z < x) = 1 - (1-x)^{kn}$

so pdf $= kn(1-x)^{kn-1}$ for $0 \le x \le 1$

otherwise it's zero

14

Problem 3. Let X be a random variable with density f and cdf F. The hazard function $h(t)$ is defined by

$$h(t) = \lim_{\delta \to 0+} \frac{1}{\delta} P(t \le X < t + \delta \,|\, X \ge t).$$

(a) (7%) State **and prove** (in detail) a simple general formula for $h(t)$.

$$h(t) = \lim_{\delta \to 0^+} \frac{1}{\delta} P(t \le X < t+\delta \,|\, X \ge t)$$

$$P(t \le X < t+\delta \,|\, X \ge t)$$

$$= \frac{P(t \le t \le t+\delta)}{P(X \ge t)} = \frac{F(t+\delta) - F(t)}{1 - F(t)}$$

$$\text{so} \quad \lim_{\delta \to 0^+} \frac{1}{\delta} \frac{F(t+\delta) - F(t)}{1 - F(t)} = \lim_{\delta \to 0^+} \frac{F(t+\delta) - F(t)}{\delta} \cdot \frac{1}{1 - F(t)}$$

$$= \frac{f(t)}{1 - F(t)}$$

(b) (7%) Find $h(t)$ when $X \sim \text{Exponential}(\beta)$. (You may use the formula derived in the previous part or work directly from the definition of h.)

$$X \sim \text{Exponential}(\beta) \qquad f(x) = \frac{1}{\beta} e^{-x/\beta}$$

$$F(x) = \int_0^x \frac{1}{\beta} e^{-x/\beta} \, dx$$

$$= -e^{-x/\beta} \Big|_0^x = 1 - e^{-x/\beta}$$

$$\text{so} \quad h(t) = \frac{\frac{1}{\beta} e^{-x/\beta}}{e^{-x/\beta}} = \frac{1}{\beta}$$

14

Problem 4. (15%) The kurtosis of a random variable X is defined to be

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} \quad \text{where} \quad \mu_k = E[(X - EX)^k].$$

Evaluate α_4 when X has density

$$f(x) = 1 - |x| \quad \text{for} \quad -1 < x < 1.$$

$$EX = \int_{-1}^{1} x f(x)\,dx = \int_{-1}^{0} x(1+x)\,dx + \int_{0}^{1} x(1-x)\,dx$$

$$= \int_{-1}^{0} (x + x^2)\,dx + \int_0^1 (x - x^2)\,dx$$

$$= \frac{1}{2}x^2 + \frac{1}{3}x^3 \Big|_{-1}^{0} + \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1$$

$$= -\frac{1}{2} + \frac{1}{3} + \frac{1}{2} - \frac{1}{3}$$

$$= 0 \checkmark$$

so

$$\mu_2 = EX^2 = \int_{-1}^{1} x^2 (1-|x|)\,dx$$

$$= \int_{-1}^{0} x^2(1+x)\,dx + \int_0^1 x^2(1-x)\,dx$$

$$= \int_{-1}^{0} x^2\,dx + \int_{-1}^{0} x^3\,dx + \int_0^1 x^2\,dx - \int_0^1 x^3\,dx$$

$$= 2\left(\int_{-1}^{0} x^2\,dx + \int_{-1}^{0} x^3\,dx\right) = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

$$\mu_4 = Ex^4 = \int_{-1}^{1} x^4 (1 - |x|)\,dx$$

$$= 2\int_{-1}^{0} x^4(1+x)\,dx = 2\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{15} \checkmark$$

$$a_4 = \frac{\mu_4}{\mu_2^2} = \frac{1}{15} \Big/ \left(\frac{1}{6}\right)^2 = \frac{36}{15} = \frac{12}{5}$$

15

Problem 5. Five flies are trapped in a car and occupy themselves by biting the driver who attempts to kill them with a flyswatter. Assume that the flies' lifetimes (which are ended by the flyswatter) are i.i.d. exponentially distributed with a mean of β hours, and that, while k flies remain, they bite the driver at an average rate of ck^2 bites per hour. (β and c are arbitrary positive values.)

(a) (8%) What is the expected value of the total number of bites the driver will receive?

$X_{(i)} = \{$ time from 0 to the i-th fly dies $\}$

We can divide the line intervial into

$X_{(1)}$, $X_{(2)} - X_{(1)}$, ... $X_{(3)} - X_{(2)}$, $X_{(4)} - X_{(3)}$, $X_{(5)} - X_{(4)}$

$X_{(1)} \sim$ exponential $(\frac{\beta}{5})$ $\quad X_{(2)} - X_{(1)} \sim$ exponential $(\frac{\beta}{4})$

$X_{(3)} - X_{(2)} \sim$ exponential $(\frac{\beta}{3})$ $\quad X_{(4)} - X_{(3)} \sim$ exponential $(\frac{\beta}{2})$

$X_{(5)} - X_{(4)} \sim$ exponential (β) and they are independent.

$$E(\text{total bites}) = E\Big(5^2c \cdot X_{(1)} + 4^2c \cdot (X_{(2)} - X_{(1)}) + 3^2c(X_{(3)} - X_{(2)}) + 2^2c \cdot (X_{(4)} - X_{(3)}) + c\,(X_{(5)} - X_{(4)})\Big)$$

(b) (6%) What is the **variance** of the time until the last fly dies?

From part a.

$$Var(X_{(5)}) = Var\Big(X_{(1)} + (X_{(2)} - X_{(1)}) + (X_{(3)} - X_{(2)}) + (X_{(4)} - X_{(3)}) + (X_{(5)} - X_{(4)})\Big)$$

and they are independ. and we know the variance of each of these

$$= \frac{\beta^2}{25} + \frac{\beta^2}{16} + \frac{\beta^2}{9} + \frac{\beta^2}{4} + \beta^2$$

cont Part a.

$$= 25c \cdot E(X_{(1)}) + 16c\,E(X_{(2)} - X_{(1)}) + 9c\,E(X_{(3)} - X_{(2)}) + 4c\,E(X_{(4)} - X_{(3)}) + c\,E(X_{(5)} - X_{(4)})$$

$$= 25c \cdot \frac{\beta}{5} + 16 \cdot c\frac{\beta}{4} + 9 \cdot c \cdot \frac{\beta}{3} + 4 \cdot c\frac{\beta}{2} + \beta$$

$$= (5 + 4 + 3 + 2 + 1)\beta c$$

$$= 15\beta c$$

14

Problem 6. (10%) Suppose X and Y are independent and normally distributed (written $N(\mu, \sigma^2)$ with

$$X \sim N(3,5) \quad \text{and} \quad Y \sim N(6,2)$$

What is the moment generating function (mgf) of $W = 2X - 3Y + 5$? (Write it out explicitly.)

$$M_W(t) = E\left(e^{(2X-3Y+5)t}\right)$$

$$= E\left(e^{2t\cdot X - 3tY + 5t}\right)$$

Since X, Y independent $= e^{5t} M_X(2t) \cdot M_Y(-3t)$

$X \sim N(3,5)$ so $M_X(t) = e^{3t + \frac{5}{2}t^2}$

$Y \sim N(6,2)$ so $M_Y(t) = e^{6t + t^2}$

so $M_X(2t) = e^{3(2t) + \frac{5}{2}\cdot 4t^2} = e^{6t + 10t^2}$

$M_Y(-3t) = e^{6(-3t) + (3t)^2} = e^{-18t + 9t^2}$

so $M_W(t) = e^{5t} \cdot e^{6t + 10t^2} \cdot e^{-18t + 9t^2}$

$= e^{-7t + 19t^2}$

10

No work is required for the remaining problems. You will receive full credit for stating the correct answer.

Problem 7. A class contains M males and F females. An ordered sample of size k is drawn from the class (that is, students are sampled one-by-one without replacement).

(a) (5%) What is the **number** of ordered samples in which the 3[rd] person is a male?

$$\binom{M}{1}\binom{M+F-1}{k-1}(k-1)!$$

(b) (4%) What is the **probability** that the 3[rd] and k^{th} (last) persons are both males?

$$\frac{M}{M+F}\cdot\frac{M-1}{M+F-1}$$

(c) (5%) What is the probability the sample contains exactly 3 males?

$$\frac{\binom{M}{3}\binom{F}{k-3}}{\binom{M+F}{k}}$$

Problem 8. (4%) Suppose X_1 and X_2 are independent with $X_i \sim$ Negative Binomial(r_i, p_i) for $i = 1, 2$. Then $X_1 + X_2$ will have a negative binomial distribution if ... (Circle the one correct response.)

a) $r_1 = r_2$	b) $r_1 \neq r_2$	c) $r_1 = r_2 = 1$	d) $r_1, r_2 > 0$
e) $p_1, p_2 > 1$	f) $p_1, p_2 < 0$	g) $p_1 = p_2$	h) $p_1 \neq p_2$

(14+4)