

Please read the following directions.

DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the “Table of Common Distributions” given in the back of the text.)
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don’t get carried away! Show enough work so that what you have done is clearly understandable.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No “cooperation” is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** pages.
- There are a total of **100** points.

Problem 1.

(a) (4%) Give a definition for the gamma **function**, $\Gamma(\alpha)$, for $\alpha > 0$.

See notes4.pdf, page 21.

$$\Gamma(\alpha) = \underline{\hspace{10cm}}$$

(b) (11%) Suppose X has a gamma **distribution**, $X \sim \text{Gamma}(\alpha, \beta)$. **Without** using mgf's, derive a general formula for EX^k , the k^{th} moment of X .

See exercise 3.17 and notes4.pdf, page 22.

Problem 2. Let X_1, X_2, \dots, X_n be i.i.d. random variables with cumulative distribution function (**cdf**) $F(x) = 1 - (1 - x)^k$ for $0 \leq x \leq 1$.

See exercise C3 and notes7.pdf, page 7

(a) (7%) Define $Y = \max X_i$. Find the **density** (**pdf**) of Y .

(b) (7%) Define $Z = \min X_i$. Find the **density** (**pdf**) of Z .

Problem 3. Let X be a random variable with density f and cdf F . The hazard function $h(t)$ is defined by

$$h(t) = \lim_{\delta \rightarrow 0+} \frac{1}{\delta} P(t \leq X < t + \delta \mid X \geq t).$$

(a) (7%) State **and prove** (in detail) a simple general formula for $h(t)$.

See exercise 3.25. (Students should lose points for replicating the typo in the solution.)

(b) (7%) Find $h(t)$ when $X \sim \text{Exponential}(\beta)$. (You may use the formula derived in the previous part or work directly from the definition of h .)

See exercise 3.26 and notes7.pdf, page 5.

Problem 4. (15%) The kurtosis of a random variable X is defined to be

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} \quad \text{where} \quad \mu_k = E[(X - EX)^k].$$

Evaluate α_4 when X has density

$$f(x) = 1 - |x| \quad \text{for} \quad -1 < x < 1.$$

See exercise 2.28.

Problem 5. Five flies are trapped in a car and occupy themselves by biting the driver who attempts to kill them with a flyswatter. Assume that the flies' lifetimes (which are ended by the flyswatter) are i.i.d. exponentially distributed with a mean of β hours, and that, while k flies remain, they bite the driver at an average rate of ck^2 bites per hour. (β and c are arbitrary positive values.)

(a) (8%) What is the expected value of the total number of bites the driver will receive?

See exercise C4.

(b) (6%) What is the **variance** of the time until the last fly dies?

See notes7.pdf, page 12.

Problem 6. (10%) Suppose X and Y are independent and normally distributed (written $N(\mu, \sigma^2)$) with

$$X \sim N(3, 5) \quad \text{and} \quad Y \sim N(6, 2)$$

What is the moment generating function (mgf) of $W = 2X - 3Y + 5$? (Write it out explicitly.)

This may be worked out by manipulating mgf's as in the lecture notes in notes5.pdf, pages 17-18 or by plugging into this result and using the expression for the mgf of the normal distribution given in the appendix.

No work is required for the remaining problems. You will receive full credit for stating the correct answer.

Problem 7. A class contains M males and F females. An ordered sample of size k is drawn from the class (that is, students are sampled one-by-one without replacement).

See notes6.pdf, page 22

- (a) (5%) What is the **number** of ordered samples in which the 3rd person is a male?
- (b) (4%) What is the **probability** that the 3rd and k^{th} (last) persons are both males?
- (c) (5%) What is the probability the sample contains exactly 3 males?

The number of males in the sample has a hypergeometric distribution. Use the pmf for the hypergeometric, or re-derive it by counting arguments.

Problem 8. (4%) Suppose X_1 and X_2 are independent with $X_i \sim \text{Negative Binomial}(r_i, p_i)$ for $i = 1, 2$. Then $X_1 + X_2$ will have a negative binomial distribution if ... (Circle the one correct response.)

The closure property for the negative binomial distribution requires $p_1 = p_2$.

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|-------------------|-------------------|--------------------|-------------------|
| a) $r_1 = r_2$ | b) $r_1 \neq r_2$ | c) $r_1 = r_2 = 1$ | d) $r_1, r_2 > 0$ |
| e) $p_1, p_2 > 1$ | f) $p_1, p_2 < 0$ | g) $p_1 = p_2$ | h) $p_1 \neq p_2$ |