TEST #3
STA 5326
December 1, 2010

Name:______________________________________

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- The different problems are not related.
- The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- Show and explain your work (including your calculations) for all problems unless you are explicitly told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.
- There are a total of **100** points.

The Bivariate Normal Density:

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_x}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)\right)$$

Problem 1. Suppose X and Y have the joint density

$$f(x,y) = \frac{1}{8\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{8(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right) \quad \text{for } -\infty < x < \infty, -\infty < y < \infty$$

(a) (3%) What is the marginal distribution of X? State the name of the distribution and the values of any parameters. (No work is required for this part.)

Comparison with the bivariate normal joint density given on the cover page shows that density $f(x,y)$ above is a bivariate normal with $\mu_X = \mu_Y = 0$ and $\sigma_X = \sigma_Y = 2$. Therefore, using properties of the bivariate normal from lecture, we know that $X \sim N(0, \sigma^2 = 4)$.

the marginal distribution of $\boldsymbol{X}$ is ____

(b) (9%) Verify your answer by calculating the marginal density of X from the joint density given above.

This is part of exercise 4.45, but simplified by setting $\mu_X = \mu_Y = 0$.

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\
&= \frac{1}{8\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2}{8(1-\rho^2)}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{8(1-\rho^2)}\left(-2\rho xy + y^2\right)\right) dy \\
&= \frac{1}{8\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2}{8(1-\rho^2)}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{8(1-\rho^2)}\left((y-\rho x)^2 - \rho^2 x^2\right)\right) dy \\
&= \frac{1}{8\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2}{8(1-\rho^2)}\right) \exp\left(\frac{\rho^2 x^2}{8(1-\rho^2)}\right) \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{(y-\rho x)^2}{8(1-\rho^2)}\right) dy \\
&= \frac{1}{8\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2}{8}\right) \cdot \sqrt{2\pi \cdot 4(1-\rho^2)} \\
&= \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{8}\right)
\end{aligned}$$

$\boldsymbol{f_X(x)} =$ ____

(c) (9%) Calculate the conditional density of Y given $X = x$.

This is also part of exercise 4.45.

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\
&= \frac{\dfrac{1}{8\pi\sqrt{1-\rho^2}} \exp\left(-\dfrac{1}{8(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right)}{\dfrac{1}{\sqrt{8\pi}} \exp\left(-\dfrac{x^2}{8}\right)} \\
&= \frac{1}{\sqrt{8\pi(1-\rho^2)}} \exp\left(-\frac{1}{8(1-\rho^2)}\left(x^2 - (1-\rho^2)x^2 - 2\rho xy + y^2\right)\right) \\
&= \frac{1}{\sqrt{8\pi(1-\rho^2)}} \exp\left(-\frac{1}{8(1-\rho^2)}\left(\rho^2 x^2 - 2\rho xy + y^2\right)\right) \\
&= \frac{1}{\sqrt{8\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{8(1-\rho^2)}\right)
\end{aligned}
$$

It is possible to answer this part without having done part (b) and obtaining the marginal density $f_X(x)$. Just note that $f_X(x)$ in the denominator of $f_{Y|X}(y|x)$ does not depend on y and so may be regarded as a constant (part of the normalizing constant). As a function of y for a fixed value of x, the conditional density $f_{Y|X}(y|x)$ is simply proportional to $f(x,y)$, which is easily seen to be proportional to the answer we obtained above. Here are the details.

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\
&\propto f(x,y) = \frac{1}{8\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{8(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right) \\
&\propto \exp\left(-\frac{1}{8(1-\rho^2)}\left(-2\rho xy + y^2\right)\right) \\
&\propto \exp\left(-\frac{1}{8(1-\rho^2)}\left(\rho^2 x^2 - 2\rho xy + y^2\right)\right) \\
&\propto \frac{1}{\sqrt{8\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{8(1-\rho^2)}\right)
\end{aligned}
$$

$\boldsymbol{f_{Y|X}(y|x)} =$ ____

(d) (3%) What is the conditional distribution of Y given $X = x$? Name the distribution whose density you found above, and give the values of any parameters. (No work is required.)

From the answer to part (c) or from the lecture notes, we know that the conditional distribution is $N\left(\mu = \rho x, \sigma^2 = 4(1-\rho^2)\right)$ See the lecture notes on page 9 of `notes12.pdf`.

cond. distn. of $\boldsymbol{Y}$ given $\boldsymbol{X = x}$ is ____

Problem 2. (20%) Let (X, Y) be a bivariate random vector with joint pdf $f(x, y)$. Let

$$S = X + aY + b\,, \quad T = cY + k$$

where a, b, c, k are fixed constants. Find an expression for the joint pdf of (S, T).

This is similar to exercise 4.22.

$\boldsymbol{f_{S,T}(s,t)} =$ ____

Problem 3. Let Y be the number of seeds in a randomly chosen packet. Suppose Y has a Poisson distribution with mean λ. Each seed, when planted, has probability p of growing and producing a plant. Suppose a packet is chosen at random and all the seeds in it are planted. Let X be the number of plants which then grow.

This is similar to the example on page 16 of notes11.pdf, and to exercises 4.31 and 4.32(a).

(a) (9%) Find the mean of X.

$\boldsymbol{EX} =$ ________________________________

(b) (10%) Find the variance of X.

$\mathrm{Var}(\boldsymbol{X}) =$ ________________________________

[**Problem 3 continued**]

(c) (4%) Are X and Y independent? Clearly answer "Yes" or "No" and justify your answer.

No, they are not independent. The easiest way to prove this is to show that the support is not a Cartesian product set. This is because it is always the case that $X \le Y$ since the number of plants cannot be larger than the number of seeds. This implies that the support of (X, Y) is the set of pairs of integers (x, y) satisfying $0 \le x \le y$, which is not a product set.

An alternative argument: the fact that the distribution of $Y|X$ depends on X also implies that X and Y are **not** *independent.*

Problem 4. (6%) A random point (X, Y) is distributed uniformly on the **triangle** with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. For this random point, what is $P(X > a)$ where $0 < a < 1$?

This is somewhat similar to the parts of Exercise 4.1 and to the lecture example on page 4 of notes9.pdf. The easiest way to work the problem is to note that, for a random point **uniformly** *distributed on some region, probability is proportional to area. So to calculate the probability the random point falls in some set, just divide the area of the set by the total area of the region.*

Alternatively, you can calculate probabilities by doing a double integral of the joint pdf $f(x, y)$ over the appropriate set. For a uniform distribution on a region, the joint density is $f(x, y) = c$ inside the region (and zero outside) with c chosen so that the joint pdf integrates to 1 over the entire region. For the triangle in this problem (which has area 1/2) we get $c = 2$.

$\boldsymbol{P(X > a)} =$ ____

Problem 5. Suppose X and Y are discrete random variables taking the values $1, 2, 3, 4$. Use the table below in the following two parts.

k	$E(Y\|X = k)$	$P(X = k)$	$P(Y = k)$
1	1	1/20	10/20
2	4/3	3/20	6/20
3	5/3	6/20	3/20
4	2	10/20	1/20

The random variable $E(Y|X)$ takes the values given in the column $E(Y|X = k)$ with the probabilities given in the column $P(X = k)$, i.e., these two columns specify the pmf of $E(Y|X)$. See the lecture notes on page 13 of notes9.pdf and pages 4–5 of notes11.pdf.

(a) (3%) Find $P\{E(Y|X) > 1.5\}$.

$\boldsymbol{P\{E(Y|X) > 1.5\}} =$ ________________________________

(b) (3%) Find $E[\{E(Y|X)\}^2]$.

$\boldsymbol{E[\{E(Y|X)\}^2]} =$ ________________________________

Problem 6. The transition probability matrix P for a Markov chain $X_0, X_1, X_2, \ldots$ with state space $\{1, 2, 3, 4\}$ is given below along with the matrix products P^2 and P^3.

$$P = \begin{pmatrix} 0.11 & 0.36 & 0.23 & 0.3 \\ 0.01 & 0.03 & 0.55 & 0.41 \\ 0.07 & 0.75 & 0.14 & 0.04 \\ 0.28 & 0.19 & 0.48 & 0.05 \end{pmatrix}$$
$$P^2 = \begin{pmatrix} 0.1158 & 0.2799 & 0.3995 & 0.2048 \\ 0.1547 & 0.4949 & 0.2926 & 0.0578 \\ 0.0362 & 0.1603 & 0.4674 & 0.3361 \\ 0.0803 & 0.476 & 0.2601 & 0.1836 \end{pmatrix}$$
$$P^3 = \begin{pmatrix} 0.1008 & 0.3886 & 0.3348 & 0.1757 \\ 0.0586 & 0.301 & 0.3765 & 0.2639 \\ 0.1324 & 0.4322 & 0.3233 & 0.1121 \\ 0.0832 & 0.2731 & 0.4048 & 0.2388 \end{pmatrix}$$

(a) (5%) Find the value of $P_4(X_1 = 3, X_4 = 2, X_6 = 1)$. [Note: This probability may also be written as $P(X_1 = 3, X_4 = 2, X_6 = 1 \,|\, X_0 = 4)$.] (Show a little work.)

probability=________________________________

(b) (3%) Suppose the chain starts (at time zero) in a randomly chosen state with probabilities given by the vector $a = (.5, .2, .2, .1)$. What is $P(X_{10} = 3)$, the probability the chain is in state 3 at time 10? [Use matrix notation and state your answer as a specific entry in a specific vector or matrix for which you give a formula. Do NOT attempt to evaluate your answer.] (No work is required.)

$\boldsymbol{P(X_{10} = 3)} =$________________________________

No work is required for the problems on this page. You will receive full credit just for stating (or circling) the correct answer.

Problem 7.

See page 20 of notes10.pdf.

(a) (3%) Suppose X and Y are independent random variables with densities $f_X(x)$ and $f_Y(y)$. State a general formula for the density of $Z = X + Y$. (This formula is called the convolution integral.)

$f_Z(z) =$ ____

Problem 8. (3%) Apply this formula to the case where X and Y are exponential random variables with densities $f_X(x) = \alpha e^{-\alpha x}$ for $x > 0$ and $f_Y(y) = \lambda e^{-\lambda y}$ for $y > 0$. Write an explicit integral (with explicit limits!) for the density $f_Z(z)$ of $Z = X + Y$. (Do NOT evaluate this integral.)

$f_Z(z) =$ ____

Problem 9. (4%) Suppose X is a random variable with mean μ and variance σ^2, and b is a constant. What is the value of $\operatorname{Cov}(X + b, X - b)$?

a) $\sigma^2 - b^2$ **b)** $\sigma^2 + b^2$ **c)** μ^2 **d)** σ^2 **e)** $\rho\sigma^2$
f) $(\mu + b)(\mu - b)$ **g)** $\sigma^2 - (\mu + b)(\mu - b)$ **h)** 0 **i)** 1/2 **j)** 1

Problem 10. (3%) Suppose X and Z are independent random variables with $X \sim \text{Uniform}(-2, 2)$ and

$$Z = \begin{cases} +1 & \text{with probability } 1/2, \\ -1 & \text{with probability } 1/2. \end{cases}$$

Define $Y = ZX$. What is the value of $P(|X| < 1, |Y| > 1)$?

a) 0 **b)** 1/16 **c)** 1/4 **d)** 1/2 **e)** 3/4 **f)** 1