TEST #1
STA 5326 **Name:**____________________________________
September 27, 2012

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- The different problems are not related. The different parts of a problem are usually unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all problems unless you are explicitly told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** pages.
- There are a total of **100** points.

Problem 1. Let α, β be two fixed positive constants, and define the density (pdf)

$$f(x) = \begin{cases} \dfrac{\alpha}{2} e^{\alpha x} & \text{if } x < 0\,, \\ \dfrac{\beta}{2} e^{-\beta x} & \text{if } x \geq 0\,. \end{cases}$$

Suppose X is a random variable with density f.

This problem is similar to 2.4.

(a) (10%) Find the cumulative distribution function (cdf) of X.

The answer is

$$F_X(x) = \begin{cases} \frac{1}{2} e^{\alpha x} & \textit{if } x < 0\,, \\ \frac{1}{2} + \frac{1}{2}(1 - e^{-\beta x}) & \textit{if } x \geq 0\,. \end{cases}$$

(b) (10%) Find $P(|X| < t)$ for all $t > 0$.

The answer is

$$P(|X| < t) = 1 - \frac{1}{2}(e^{-\alpha t} + e^{-\beta t}) \quad \textit{for } t > 0\,.$$

Problem 2. Suppose the random variables $Y_1, Y_2, Y_3, \ldots$ are independent with $P\{Y_i = 1\} = \boldsymbol{p}$ and $P\{Y_i = 0\} = \mathbf{1} - \boldsymbol{p}$ for all i where $0 < p < 1$. Define

$$X = \sum_{i=1}^{\infty} \frac{Y_i}{\mathbf{3}^i} = \frac{Y_1}{\mathbf{3}} + \frac{Y_2}{\mathbf{9}} + \frac{Y_3}{\mathbf{27}} + \cdots$$

Calculate the following.

This problem is similar to A3.

(a) (10%) $E(X)$

$$EX = \sum_{i=1}^{\infty} \frac{EY_i}{\mathbf{3}^i} = \sum_{i=1}^{\infty} \frac{p}{\mathbf{3}^i} = \frac{p/3}{1-(1/3)} = \frac{p}{2}.$$

(b) (10%) Var(X)

$$Var(X) = \sum_{i=1}^{\infty} \frac{Var(Y_i)}{(\mathbf{3}^i)^2} = \sum_{i=1}^{\infty} \frac{p(1-p)}{\mathbf{3}^{2i}} = \frac{p(1-p)/9}{1-(1/9)} = \frac{p(1-p)}{8}.$$

Problem 3. Suppose the monkey types **15** letters at random. (Each keystroke is independent of the others with all 26 possibilities equally likely.)

Note: Parts (b) and (c) refer to "vowels" which are the **5** letters AEIOU.

(a) (8%) What is the probability these 15 letters are in alphabetical order? (Here we are allowing repetitions so that something like BBBJJLMNOPXXXZZ is allowed.)

This part is similar to B2(b).
Here is a solution. Let $A = \{$the 15 letters are in alphabetical order$\}$. Then

$$P(A) = \frac{\#(A)}{\#(\Omega)} = \frac{\binom{26+15-1}{15}}{26^{15}} = \frac{\binom{40}{15}}{26^{15}}$$

where the numerator is obtained by an application of the counting rule in Table 1.2.1 for "Unordered, With replacement"; we are counting the number of ways to sample $r = 15$ letters from $n = 26$ when the sampling is with replacement and the order of selection does not matter (for the letters must be placed in alphabetical order). One can also justify the numerator by repeating the argument on page 15 of the text (also given in the solutions for 1.19 and the second solution of B2(b)): the numerator is the same as the number of 26-tuples of non-negative integers whose sum is 15, which is the number of ways to arrange 15 markers and 25 walls.

(b) (8%) Compute the following conditional probability: given that all the 15 typed letters are distinct (i.e., there are no repeated letters), what is the probability they contain exactly 3 vowels?

This is similar to 1.51. Because of the conditioning, this part is essentially equivalent to the following: An urn contains 26 letters of which 5 are vowels. 15 letters are selected at random from the urn. What is the probability there are exactly 3 vowels in these 15 letters?
Here is a solution. Let $B = \{$exactly 3 vowels$\}$ and $C = \{$no repeated letters$\}$. Then

$$P(B|C) = \frac{\#(B \cap C)}{\#(C)} = \frac{\binom{5}{3}\binom{21}{12}15!}{\binom{26}{15}15!} = \frac{\binom{5}{3}\binom{21}{12}}{\binom{26}{15}}.$$

The denominator $\#(C)$ is the number of ordered sequences of 15 distinct letters. Each such sequence may be obtained by choosing 15 letters from the 26 (in $\binom{26}{15}$ ways) and then arranging them in some order (in 15! ways). The numerator $\#(B \cap C)$ is the number of ordered sequences of 15 distinct letters which contain exactly 3 vowels (and therefore 12 consonants). Each such sequence may be obtained by choosing 3 vowels from 5, choosing 12 consonants from 21, and then arranging these 15 letters in some order, giving the numerator above.

[**Problem 3 continued**]

(There is no conditioning in this last part below; repetitions are allowed.)

(c) (10%) What is the probability that all 5 vowels appear at least once in the monkey's 15 typed letters? (Your answer does **not** have to be simplified.)

This is similar to 1.20; "at least one call on each of 7 days" is like "at least one occurrence of each of 5 vowels". The same sort of inclusion-exclusion solution can be used. See page 5 of solutions1_text.pdf.

Here is the solution: We switch to the complementary event and then use the principle of inclusion-exclusion.

$$\begin{aligned}
&P(\textit{at least one occurrence of each vowel}) \\
&\quad = 1 - P(\textit{at least one vowel with no occurrences}) \\
&\quad = 1 - P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5)
\end{aligned}$$

where we define $A_i = \{\textit{no occurrences of vowel } i\}$ *for* $i = 1, 2, \ldots, 5$ *and the vowels are ordered AEIOU.*

$$\begin{aligned}
&P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) \\
&\quad = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + P(A_1 \cap A_2 \cap \cdots \cap A_5) \\
&\quad = \binom{5}{1} P(A_1) - \binom{5}{2} P(A_1 \cap A_2) + \binom{5}{3} P(A_1 \cap A_2 \cap A_3) - \cdots + \binom{5}{5} P(A_1 \cap \cdots \cap A_5) \\
&\quad = \binom{5}{1}\left(\frac{25}{26}\right)^{15} - \binom{5}{2}\left(\frac{24}{26}\right)^{15} + \binom{5}{3}\left(\frac{23}{26}\right)^{15} - \binom{5}{4}\left(\frac{22}{26}\right)^{15} + \binom{5}{5}\left(\frac{21}{26}\right)^{15} \\
&\quad = \sum_{i=1}^{5} (-1)^{i-1} \binom{5}{i} \left(\frac{26-i}{26}\right)^{15}
\end{aligned}$$

So the final answer is

$$1 - \sum_{i=1}^{5} (-1)^{i-1} \binom{5}{i} \left(\frac{26-i}{26}\right)^{15}.$$

Problem 4. Let U be a Uniform$(0, 1)$ random variable, and X be a random variable with cumulative distribution function (cdf) given by

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x) \, .$$

Parts (a) and (b) of this problem are based on the discussion in pages 17–21 of notes3.pdf.

(a) (6%) Find a function h such that $h(U) \stackrel{d}{=} X$.

(b) (6%) Find a function g such that $g(X) \stackrel{d}{=} U$.

[**Problem 4 continued**]

(c) (10%) Find the density (pdf) of the random variable $Y = X^2$.

Note: $\dfrac{d}{dx}\tan^{-1}(x) = \dfrac{1}{1+x^2}$

Differentiate the cdf of X to get the pdf of X. Then you have a standard transformation problem: $Y = g(X)$ where the domain can be broken into two pieces $(-\infty, 0)$ and $(0, \infty)$ on which g is monotomic. Similar to 2.6(b) or 2.7.

Problem 5. Suppose a dart is tossed uniformly at random on a circular target of radius R. Let Z be the distance of the dart from the center of the circle.

(a) (6%) Let P_Z be the induced probability function of Z and suppose $0 \leq a \leq b \leq R$. Find $P_Z\big((a, b)\big)$.

Based on page 15 of notes2.pdf.

(b) (6%) Find the density (pdf) of Z.

Use part (a) to get the cdf. Then differentiate to get the pdf.