TEST #2
STA 5326
November 1, 2012

Name: Some Answers (solutions not complete)

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Show and explain your work (including your calculations) for all problems unless you are explicitly told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out write down this approach. If you know a useful result, write it down.)
- The different problems are not related. The different parts of a problem are usually unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **10** pages.
- There are a total of **100** points.

Problem 1. A new drug and an old drug are both equally effective, causing improvement in the patient's condition in 70% of the cases in which they are used. Suppose the new drug and old drug are each given to 200 patients.

(a) (8%) What is the exact probability that 133 or more of the patients given the new drug improve? (Give an expression for the exact probability, but do **not** try to evaluate it.)

X = # who improve on new drug.

Answer: $X \sim \text{Binomial}(n=200, p=.70)$

$$P(X \geq 133) = \sum_{i=133}^{200} \binom{200}{i} (.70)^i (.30)^{200-i}$$

Parts (a) and (b) are similar to 3.5

(b) (8%) Give an approximation to the above probability. (Your answer should be a numerical value.)

Answer: $Z = \dfrac{X - 200(.70)}{\sqrt{200(.7)(.3)}} \approx N(0,1)$

Using normal approx. with cont. correction

$$P\left(\underset{\substack{\uparrow \\ N(0,1)}}{Z} > \frac{133 - .5 - 200(.7)}{\sqrt{200(.7)(.3)}}\right)$$

$\cong 1 - \Phi(-1.157275)$

$= \Phi(1.157275) = .87642$

or $\Phi(1.16) = .8770$ using table

[Problem 1 continued] Similar to C5 and notes7.pdf pages 21–25.

(c) (8%) Find the approximate probability that the number of patients who improve with the new drug is **exactly one more** than the number who improve with the old drug. (Your answer should be a numerical value.)

$X = \#$ improve on new $\sim N(200(.7), 200(.7)(.3))$

$Y = \#$ improve on old $\sim$ same

$D = X - Y \approx N(0, 2(200)(.7)(.3))$

$P(D=1) \approx P(.5 < D^* < 1.5)$

D^*: an exactly normal rv

$= P\left(\frac{.5-0}{\sigma} < Z < \frac{1.5-0}{\sigma}\right)$, $Z \sim N(0,1)$

where $\sigma = \sqrt{2(200)(.7)(.3)} = 9.165151$

$= P(\underbrace{.05455447}_{a} < Z < \underbrace{.1636634}_{b})$

$= \Phi(b) - \Phi(a) = .04324865$

$\approx \Phi(.16) - \Phi(.05) = .5636 - .5199 = .0437$ by table

or $\approx (b-a)\varphi\left(\frac{a+b}{2}\right)$, $\varphi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$

$= .04327$ using 0 is OK too.

$(= .04353$ if 0 is used instead of $(a+b)/2)$

Problem 2. Let $A_1, A_2, \ldots, A_k$ be events and $X_1, X_2, \ldots, X_k$ be the corresponding indicator random variables (that is, $X_i = I_{A_i}$ for all i). Suppose $P(A_i) = \alpha$ for all i and $P(A_i \cap A_j) = \beta$ whenever $i \neq j$. Let $S = \sum_{i=1}^{k} X_i$. Evaluate the following.

(a) (8%) ES

$$ES = kEX_1 = kP(A_1) = k\alpha$$

(See page 21 of notes6.pdf and discussion of indicator rv's on pages 9–12 of notes4.pdf)

(b) (8%) $E(S^2)$

$$ES^2 = E\Big(\sum_{i,j} X_i X_j\Big) = \sum_{i,j} EX_iX_j$$

$$= \sum_{i,j} P(A_i \cap A_j) = \underbrace{kP(A_1)}_{\text{terms with } i=j} + \underbrace{(k^2-k)P(A_1 \cap A_2)}_{\text{terms with } i \neq j}$$

$$= k\alpha + (k^2-k)\beta$$

(See pages 24–25 of notes6.pdf.)

(c) (6%) $E(I_{A_1^c} I_{A_2^c})$

$$= E(1-I_{A_1})(1-I_{A_2})$$

$$= E(1 - I_{A_1} - I_{A_2} + I_{A_1 \cap A_2})$$

$$= 1 - P(A_1) - P(A_2) + P(A_1 \cap A_2)$$

$$= 1 - 2\alpha + \beta$$

(See p. 12 of notes4.pdf for a related example.)

Problem 3. A pdf $f(x)$ is said to be symmetric about a if $f(a+u) = f(a-u)$ for all u. Suppose the pdf of X is symmetric about a.

(a) (8%) Show that $\int_{-\infty}^{a} f(x)\,dx = 1/2$.

See 2.26(b)

[**Problem 3 continued**]

(b) (8%) Show that, if EX exists, then $EX = a$.

See 2.26 (c)

Problem 4. Let X have a discrete uniform distribution that puts equal probability on each of the integers $N_0, N_0+1, \ldots, N_1$.

(a) (8%) Find EX

See 3.1

or use the appendix by noting that if $X \sim$ Discrete Uniform(N_0, N_1) and $Y \sim$ Dis. Uniform$(1, n)$, with $n = N_1 - N_0 + 1$,

then $X \stackrel{d}{=} Y + N_0 - 1$.

Thus
$$EX = EY + N_0 - 1$$
$$= \frac{n+1}{2} + N_0 - 1$$
$$= \frac{N_1 - N_0 + 2}{2} + N_0 - 1$$
$$= \frac{N_0 + N_1}{2}$$

For part (b)

$$\operatorname{Var} X = \operatorname{Var} Y = \frac{(n+1)(n-1)}{12}$$
$$= \frac{(N_1 - N_0 + 2)(N_1 - N_0)}{12}$$

[**Problem 4 continued**]

(b) (8%) Find Var(X). $\left(\text{Note: } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\right)$

See 3.1

Also see previous page.

(c) (8%) Find the moment generating function (mgf) of X.

$$M_X(t) = \sum_{x=N_0}^{N_1} \frac{e^{tx}}{(N_1 - N_0 + 1)}$$

$$= \frac{e^{N_0 t} - e^{(N_1+1)t}}{1 - e^t} \cdot \frac{1}{(N_1 - N_0 + 1)}$$

by summing the finite geometric series.

Problem 5. (8%) Suppose X has density (pdf)

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } -1 \le x \le 1 \\ \frac{1}{4x^2} & \text{if } x < -1 \text{ or } x > 1 \end{cases}$$

Does EX exist? (Answer 'Yes' or 'No' and prove your answer.)

No.

$$E|X| = \int_{-\infty}^{\infty} \underbrace{|x| f(x)}_{\text{integrand is symmetric about } x=0} dx$$

$$= 2\int_0^\infty x f(x) dx$$

$$= 2\left[\int_0^1 \frac{x}{4} dx + \int_1^\infty x \cdot \frac{1}{4x^2} dx\right]$$

$$= \frac{1}{4} + \frac{2}{4}\int_1^\infty \frac{1}{x} dx = \frac{1}{4} + \frac{1}{2} \ln x \Big|_1^\infty = \infty$$

Problem 6. (6%) Give an argument to show that if $X \sim \text{Gamma}(\alpha, \beta)$ and α is large, then X is approximately normal.

If $Z_1, Z_2, Z_3, \ldots$ are iid Gamma(α_0, β) then $S_n = \sum_{i=1}^n Z_i \sim \text{Gamma}(n\alpha_0, \beta)$ by "closure" property for Gamma distn. For large n, S_n is approx Normal by CLT. Therefore Gamma(α, β) is approx normal for large α.

see p. 19–20 of notes7.pdf