TEST #3
STA 5326 **Name:**____________________________________
December 6, 2012

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Show and explain your work (including your calculations) for all problems unless you are explicitly told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- The different problems are not related. The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.
- There are a total of **100** points.

Problem 1. **Two** white balls and **three** black balls are distributed in two urns with **two** balls in the first urn and **three** in the second. We say the system is in state i, $i = 0, 1, 2$, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn.

(a) (10%) Calculate the transition probability matrix for this Markov chain.

This is very similar to Exercise #1 in Ross.
The positive transition probabilities are

$$P_{00} = 1/3, P_{01} = 2/3, P_{10} = 1/3, P_{11} = 1/2, P_{12} = 1/6, P_{21} = 1\,.$$

The most difficult case is P_{11}. *Being in state 1 means Urn 1 has 1W, 1B and Urn 2 has 1W, 2B. To make a* $1 \to 1$ *transition you must either draw a white ball from each urn or a black ball from each urn. This has probability* $1/2 \cdot 1/3 + 1/2 \cdot 2/3 = 1/2$.

To get full credit, the student should give the correct transition matrix, and show enough work to show how the entries were computed. The most difficult case (P_{11}) *should be explained in sufficient detail.*

[**Problem 1 continued**]

(b) (10%) Find the stationary distribution of this chain.
(If you did not answer the previous part, just call the transition probabilities P_{ij} and give a detailed description of the process you would use to find the stationary distribution.)

This is similar to Exercise #27 in Ross.
The stationary distribution is $\pi_0 = 3/10, \pi_1 = 3/5, \pi_2 = 1/10$.
This is obtained by solving the system $\pi P = \pi$ *and* $\sum \pi_i = 1$. *Written out explicitly this becomes:*

$$\begin{aligned}
\pi_0 &= \frac{1}{3}\pi_0 + \frac{1}{3}\pi_1 \\
\pi_1 &= \frac{2}{3}\pi_0 + \frac{1}{2}\pi_1 + \pi_2 \\
\pi_2 &= \frac{1}{6}\pi_1 \\
1 &= \pi_0 + \pi_1 + \pi_2
\end{aligned}$$

To get full credit, students should give the system of equations in detail, and solve these equations, showing enough work for you to follow what they do.
If students get the wrong matrix in part (a) but find the correct stationary distribution for the matrix they get, give them full credit. If the matrix is so completely wrong that the system of equations becomes trivial or nonsensical in some way but their explanation of how to get the stationary distribution is correct, give them 7 points (same as below).
If the students do not answer part (a) but give a good explanation of how to obtain the stationary distribution, give them 7 points.

Problem 2. Suppose X and Y are independent with densities

$$f_X(x) = 2x \quad \text{for } 0 < x < 1, \qquad f_Y(y) = \frac{y^4 e^{-y}}{4!} \quad \text{for } 0 < y < \infty\,.$$

Define $U = Y\sqrt{X}$ and $V = Y$.

(a) (14%) Find the joint density of (U, V). (Make sure to specify the support.)

The Jacobian of the transformation is $2U/V^2$. The joint density is

$$\frac{u^3 e^{-v}}{3!} \quad \textit{for } 0 < u < v < \infty$$

[**Problem 2 continued**]

(b) (6%) Find the marginal density of U. (Specify the support.)

Integrating out v gives the marginal for U as

$$\frac{u^3 e^{-u}}{3!} \quad \textit{for } 0 < u < \infty$$

(c) (4%) Are U and V independent? Answer "Yes" or "No" and justify your answer.

The answer is "No". The easiest way to see this is to note that the support of the joint density is $0 < u < v < \infty$ (everything above the line $u = v$ in the first quadrant), which is NOT a Cartesian product set.
You can also prove they are NOT independent by finding showing that the joint density of (U, V) is NOT the product of the marginal densities of U and V.

Give students 2 points for the correct answer (Yes or No) and an additional 2 points if they give a correct explanation.

Problem 3. (12%) Let $X \sim \text{Poisson}(\alpha)$ and $Y \sim \text{Poisson}(\beta)$, with X and Y independent. Calculate $P(X = k | X + Y = n)$ where $0 \leq k \leq n$.

This is problem 4.15. See the posted solutions.

Problem 4. Suppose $X|P \sim$ Negative Binomial(r, P) and $P \sim$ Beta(α, β) where $\alpha > 2$. (In your answers below, you do NOT have to simplify the expressions involving gamma functions.)

This is problem 4.34(b). See the posted solutions.

Students may leave gamma functions in their answer without simplifying them. They can also leave the Beta function $B(a, b)$ in their answers, if they give the formula for $B(a, b)$ in terms of gamma functions. They should not leave any integrals in their answer.

(a) (10%) Find EX.

(b) (10%) Find $\mathrm{Var}(X)$.

Beware: The final answer in the solution manual is wrong. The correct answer is posted in mordor.

Problem 5. Suppose $N(\cdot)$ is a non-homogeneous Poisson process with rate function $\lambda(t) = t^2$.

See page 5 of notes14.pdf.

(a) (6%) What is the distribution of $N(2) - N(1)$? (Give the name of the distribution and specify the value(s) of any parameter(s).)

$$N(2) - N(1) \sim \mathit{Poisson}\left(\int_1^2 t^2\, dt\right) = \mathit{Poisson}\left(\frac{2^3}{3} - \frac{1^3}{3}\right) = \mathit{Poisson}\left(\frac{7}{3}\right)$$

(b) (4%) What is the approximate probability of an arrival in the time interval $(3, 3 + 10^{-6})$?

Using the approximation $P(N(t + dt) - N(t) = 1) = \lambda(t)\, dt$ *with* $t = 3$ *and* $dt = 10^{-6}$ *gives probability* $= 3^2 \cdot 10^{-6} = 9 \times 10^{-6}$.

Problem 6. Suppose $\mathrm{Var}(X) = \mathrm{Var}(Y) = \alpha$ and $\mathrm{Cov}(X, Y) = \beta$.

(a) (4%) Find the correlation ρ_{UV} between $U = 2X + 3$ and $V = 5Y + 7$.

$$\begin{aligned} Cov(U,V) &= Cov(2X+3, 5Y+5) = Cov(2X, 5Y) = 2 \cdot 5\, Cov(X,Y) = 10\beta \\ Var(U) &= Var(2X+3) = Var(2X) = 4\, Var(X) = 4\alpha \\ Var(V) &= Var(5Y+7) = Var(5Y) = 25\, Var(Y) = 25\alpha \\ \rho_{UV} &= \frac{Cov(U,V)}{\sqrt{Var(U)\, Var(V)}} = \frac{10\beta}{\sqrt{4\alpha \cdot 25\alpha}} = \frac{\beta}{\alpha} \end{aligned}$$

The student's solution does NOT have to be this detailed to receive full credit.

A shorter solution is obtained by using a fact stated in lecture:

$$Correlation(aX+b, cX+d) = Correlation(X,Y)$$
$$\text{for any constants } a, b, c, d \text{ with } a > 0, c > 0.$$

This leads to

$$\begin{aligned} Correlation(2X+3, 5Y+7) &= Correlation(X,Y) \\ &= \frac{Cov(X,Y)}{\sqrt{Var(X)\, Var(Y)}} = \frac{\beta}{\sqrt{\alpha \cdot \alpha}} = \frac{\beta}{\alpha} \end{aligned}$$

(b) (4%) Find $\mathrm{Cov}(X + Y, X - Y)$.

$$\begin{aligned} Cov(X+Y, X-Y) &= Cov(X,X) - Cov(X,Y) + Cov(Y,X) - Cov(Y,Y) \\ &= Var(X) - Cov(X,Y) + Cov(X,Y) - Var(Y) \\ &= \alpha - \beta + \beta - \alpha = 0 \end{aligned}$$

The student's argument does NOT have to be this detailed to receive full credit. An argument that expands the covariance into four terms that cancel is sufficient.

Problem 7. (6%) Suppose X and Y are discrete random variables whose joint pmf is described by the following table:

	$X=0$	$X=1$	$X=2$
$Y=0$	$1/4$	$1/6$	$1/12$
$Y=1$	$1/12$	$1/12$	$1/12$
$Y=2$	$1/24$	$1/12$	$1/8$

Find $\mathrm{Var}(X|Y = 0)$.

The conditional pmf of X given $Y = 0$ is obtained by extracting the row of the table for $Y = 0$ and dividing these values by their sum (normalizing the row), yielding

x	0	1	2
$f_{X\|Y}(x\|0)$	$1/2$	$1/3$	$1/6$

The conditional mean $E(X|Y = 0)$ is the mean of the conditional pmf which is

$$E(X|Y=0) = 0 \cdot 1/2 + 1 \cdot 1/3 + 2 \cdot 1/6 = 2/3\,.$$

The conditional variance is the variance of the conditional pmf which is

$$\begin{aligned} Var(X|Y=0) &= E(X^2|Y=0) - [E(X|Y=0)]^2 \\ &= 0^2 \cdot 1/2 + 1^2 \cdot 1/3 + 2^2 \cdot 1/6 - (2/3)^2 = 1 - (4/9) = 5/9\,. \end{aligned}$$