Problem 1. (11%) Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. For a fixed number x_0, define the function

$$g(x) = \begin{cases} f(x)/F(x_0) & \text{for } x \le x_0 \\ 0 & \text{for } x > x_0. \end{cases}$$

/100

Prove that $g(x)$ is a pdf. (Assume that $F(x_0) > 0$.)
In your solution you should carefully state all the properties of a pdf.

We must show $\int_{\mathbb{R}} g(x)\,dx = 1$ and $g(x) \geq 0, \ \forall x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{x_0} \frac{f(x)}{F(x_0)}\,dx = \frac{1}{F(x_0)} \int_{-\infty}^{x_0} f(x)\,dx$$

$$= \frac{1}{F(x_0)} P(x \le x_0) = \frac{1}{F(x_0)} F(x_0) = 1$$

as needed

Also $f(x)$ is a pdf, so $f(x) \geq 0$ $(\forall x \in \mathbb{R})$ and it's given that $F(x_0) > 0$, thus if $x \le x_0$, $g(x) = \frac{f(x)}{F(x_0)} \geq \frac{0}{F(x_0)} = 0$ and if $x > x_0$, $g(x) = 0 \geq 0$. Therefore $g(x)$ satisfies the conditions of a pdf

Problem 2. (9%) Suppose X is a discrete random variable with pmf

$$f_X(x) = \begin{cases} 2^{-x} & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Find the pmf of $Y = \sqrt{X}$.

$Y = g(X) = \sqrt{X}$

$g^{-1}(y) = y^2 \Rightarrow f_Y(y) = f_X(g^{-1}(y)) = 2^{-y^2}$

$$f_Y(y) = \begin{cases} 2^{-y^2} & \text{if } y = 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots \\ & (\text{i.e. } y \in \{\sqrt{n} \mid n \in \mathbb{Z}^+\}) \\ 0 & \text{otherwise} \end{cases}$$

Problem 3. (11%) Three prisoners, A, B, and C, are on death row. The governor decides to pardon one of the three and chooses at random the prisoner to pardon. He informs the warden of his choice but requests that the name be kept secret for a few days.

The next day, A tries to get the warden to tell him who had been pardoned. The warden refuses. A then asks the warden to tell him which of B or C will be executed. The warden thinks for a while, then tells A that B is to be executed. Given this information, what is the probability that **C will be pardoned**?

(Assume that, if A is to be pardoned, the warden says "B will die" or "C will die" with equal probability.)

Let W be the event that the warden says B will be execut.

Let A, B, C be the events that the corresponding prisoner will be pardoned

We need to find $P(C|W)$

$$P(C|W) = \frac{P(W|C)P(C)}{P(W|C)P(C) + P(W|B)P(B) + P(W|A)P(A)}$$

$$= \frac{(1)(\frac{1}{3})}{(1)(\frac{1}{3}) + (0)(\frac{1}{3}) + (\frac{1}{2})(\frac{1}{3})} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

(Assuming that the warden isn't lying!).

Problem 4. (11%) Suppose A, B, C, D are any events. Use the principle of inclusion-exclusion to obtain a general expression for $P(AB \cup BC \cup CD \cup AD)$.
(Here $AB = A \cap B$, etc.)

$$P(AB \cup BC \cup CD \cup AD) = P(AB) + P(BC) + P(CD) + P(AD)$$

$$- \cancel{P(ABC)} - P(ABCD) - \cancel{P(ABD)}$$

$$- \cancel{P(BCD)} - P(BCAD) - \cancel{P(CDA)}$$

$$+ P(ABCD) + P(ABCD) + P(ABCD) + P(BCDA)$$

$$- P(ABCD)$$

$$= P(AB) + P(BC) + P(CD) + P(AD)$$

$$- P(ABC) - P(ABD) - P(ACD) - P(BCD)$$

$$+ P(ABCD)$$

Problem 5. A closet contains n pairs of shoes. Suppose n shoes are chosen at random.

(a) (11%) What is the probability that there are no matching pairs in the sample?

$$P(\text{no matching pairs}) = P(\text{exactly one chosen from each pair})$$

$$\#A = (\#\text{ choices per pair})^{(\#\text{ pairs})} = 2^n$$

$$\#\Omega = \binom{2n}{n}$$

$$P(\text{no matching pairs}) = \frac{2^n}{\binom{2n}{n}}$$

[Problem 5 continued]

(b) (6%) What is the probability that there are exactly one pair of matching shoes in the sample?

If one matching pair is chosen, there is one pair for which zero are chosen, and $(n-2)$ pairs for which exactly one shoe is chosen

Then $\#A = $ (# choices for chosen pair) · (# choices for zero pair) · (# ways to chose 1 from rem. $n-2$ pairs)

$$= n(n-1)\cdot 2^{n-2}$$

$$P(\text{exactly one matching pair}) = \frac{n(n-1)2^{n-2}}{\binom{2n}{n}}$$

Problem 6. Suppose you have a fair coin with the sides labeled $+1$ and -1. Toss this coin 3 times and let X_i be the value observed on the i-th toss. Define $X_4 = X_1 X_2 X_3$. For $i = 1, 2, 3, 4$, define A_i to be the event that $X_i = 1$.

(a) (6%) Describe (in general) what must be done to show that four events A, B, C, D are mutually independent.

We would have to show that the probability of the intersection of any combination of the four events is equal to the product of the same combination of events taken individually. That is:

$$P(AB) = P(A)P(B)$$
$$P(AC) = P(A)P(C)$$
$$P(AD) = P(A)P(D)$$
$$P(BC) = P(B)P(C)$$
$$P(BD) = P(B)P(D)$$
$$P(CD) = P(C)P(D)$$
$$P(ABC) = P(A)P(B)P(C)$$
$$P(ABD) = P(A)P(B)P(D)$$
$$P(ACD) = P(A)P(C)P(D)$$
$$P(BCD) = P(B)P(C)P(D)$$

and

$$P(ABCD) = P(A)P(B)P(C)P(D)$$

[Problem 6 continued]

(b) (10%) Show that the events A_1, A_2, A_3, A_4 are *not* mutually independent.

To show they are not mutually independent, we need only show that one of the equalities on the previous page is untrue, and we will show this for the last one

$$P(A_4) = P(X_4 = 1) = P(X_1 X_2 X_3 = 1) = P(\text{an even \# of flips are } -1)$$

$$= \frac{\binom{3}{2} + \binom{3}{0}}{2^3} = \frac{3+1}{8} = \frac{1}{2}$$

Then $P(A_1)P(A_2)P(A_3)P(A_4) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

$$P(A_1 A_2 A_3 A_4) = P(A_4 \mid A_1 A_2 A_3)\, P(A_1 A_2 A_3)$$

$$= P(X_4 = X_1 X_2 X_3 = 1 \mid X_1 = 1, X_2 = 1, \text{and } X_3 = 1) \cdot \left(\tfrac{1}{2}\right)^3$$

$$= (1)\left(\tfrac{1}{2}\right)^3 = \frac{1}{8} \neq \frac{1}{16}$$

Thus $P(A_1)P(A_2)P(A_3)P(A_4) \neq P(A_1 A_2 A_3 A_4)$ and the events are not mutually independent

Problem 7. (12%) Suppose X is a random variable with cdf

$$F_X(x) = \begin{cases} 1 - \frac{1}{x^2} & \text{for } x \geq 1 \\ 0 & \text{for } x < 1. \end{cases}$$

→ X only takes positive values

Find the cdf $F_Y(y)$ of the random variable $Y = \frac{10}{X}$. → Y also only takes pos. values

$$F_Y(y) = P(Y \leq y) = P\left(\frac{10}{X} \leq y\right)$$

$$= P\left(\frac{X}{10} \geq \frac{1}{y}\right) = P\left(X \geq \frac{10}{y}\right)$$

$$= 1 - P\left(X \leq \frac{10}{y}\right) = 1 - F_X\left(\frac{10}{y}\right)$$

$$= 1 - \left(1 - \left(\frac{y}{10}\right)^2\right) \text{ for } \frac{10}{y} \geq 1 \Rightarrow 10 \geq y > 0$$

$$F_Y(y) = \begin{cases} \frac{y^2}{100} & \text{if } 0 < y \leq 10 \\ 0 & \text{if } y \leq 0 \\ 1 & \text{if } y > 10 \end{cases}$$