**TEST #1**
**STA 5326** **Name:**____________________________________
**September 26, 2013**

**Please read the following directions.**
**DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO**

# Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- The different problems are not related. The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all problems unless you are explicitly told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **11** pages.
- There are a total of **100** points.

**Problem 1.** (11%) Let $X$ be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. For a fixed number $x_0$, define the function

$$g(x) = \begin{cases} f(x)/F(x_0) & \text{for } x \leq x_0 \\ 0 & \text{for } x > x_0. \end{cases}$$

Prove that $g(x)$ is a pdf. (Assume that $F(x_0) > 0$.)
In your solution you should carefully state all the properties of a pdf.

*This is similar to exercise 1.52.*

**Problem 2.** (9%) Suppose $X$ is a discrete random variable with pmf

$$f_X(x) = \begin{cases} 2^{-x} & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Find the pmf of $Y = \sqrt{X}$.

*This is similar to exercise 2.3.*

**Problem 3.** (11%) Three prisoners, $A$, $B$, and $C$, are on death row. The governor decides to pardon one of the three and chooses at random the prisoner to pardon. He informs the warden of his choice but requests that the name be kept secret for a few days.

The next day, $A$ tries to get the warden to tell him who had been pardoned. The warden refuses. $A$ then asks the warden to tell him which of $B$ or $C$ will be executed. The warden thinks for a while, then tells $A$ that $B$ is to be executed. Given this information, what is the probability that **$C$ will be pardoned**?

(Assume that, if $A$ is to be pardoned, the warden says "$B$ will die" or "$C$ will die" with equal probability.)

*Similar to Example 1.3.4 and exercise 1.37.*

**Problem 4.** (11%) Suppose $A$, $B$, $C$, $D$ are any events. Use the principle of inclusion-exclusion to obtain a general expression for $P(AB \cup BC \cup CD \cup AD)$.
(Here $AB = A \cap B$, etc.)

*Similar to page 5 of notes2.pdf.*

**Problem 5.** A closet contains $n$ pairs of shoes. Suppose $n$ shoes are chosen at random.

**(a)** (11%) What is the probability that there are no matching pairs in the sample?

*This is a special case of 1.21. The answer is* $2^n / \binom{2n}{n}$.

[ **Problem 5 continued** ]

**(b)** (6%) What is the probability that there are exactly one pair of matching shoes in the sample?

*This combines features of exercises 1.21 and 1.18. Constructing a selection of* $n$ *shoes which contain exactly one matching pair can be done in the following three steps: select one of the* $n$ *matching pairs and take both shoes from this pair, then select another pair from which you take no (zero) shoes, and then select one shoe from each of the remaining* $n-2$ *pairs. The first step can be performed in* $n$ *ways, the second in* $n-1$ *ways, and the third in* $2^{n-2}$ *ways (since there are two possibile choices–left shoe or right shoe–for each of the remaining* $n-2$ *pairs). So the number of choices of* $n$ *shoes with exactly one matching pair is* $n(n-1)2^{n-2}$*. There are* $\#(\Omega) = \binom{2n}{n}$ *equally likely ways to choose* $n$ *shoes from* $2n$*, so the final answer is* $\dfrac{n(n-1)2^{n-2}}{\binom{2n}{n}}$.

**Problem 6.** Suppose you have a fair coin with the sides labeled $+1$ and $-1$. Toss this coin 3 times and let $X_i$ be the value observed on the $i$-th toss. Define $X_4 = X_1 X_2 X_3$. For $i = 1, 2, 3, 4$, define $A_i$ to be the event that $X_i = 1$.

*This is exercise B8.*

**(a)** (6%) Describe (in general) what must be done to show that four events $A, B, C, D$ are mutually independent.

[ **Problem 6 continued** ]

**(b)** (10%) Show that the events $A_1, A_2, A_3, A_4$ are *not* mutually independent.

**Problem 7.** (12%) Suppose $X$ is a random variable with cdf

$$F_X(x) = \begin{cases} 1 - \dfrac{1}{x^2} & \text{for } x \geq 1 \\ 0 & \text{for } x < 1. \end{cases}$$

Find the cdf $F_Y(y)$ of the random variable $Y = \dfrac{10}{X}$.

*This is similar to part of 1.53.*

**In the next three questions, circle the single correct choice.**

*The correct answer to each question is marked with a star ($\star$).*

**Problem 8.** (3%) We say that events $A_1$ and $A_2$ are conditionally independent given $B$ if ...

*This comes from pages 6 and 7 of notes2.pdf.*

**a)** $P(B \mid A_1 \cap A_2) = P(B \mid A_1)P(B \mid A_2)$
**b)** $P((A_1 \mid B) \cap (A_2 \mid B)) = P(A_1 \mid B)P(A_2 \mid B)$
**c)**$\star$ $P(A_1 \cap A_2 \mid B) = P(A_1 \mid B)P(A_2 \mid B)$
**d)** $P((B \mid A_1) \cap (B \mid A_2)) = P(B \mid A_1)P(B \mid A_2)$
**e)** $P(A_1 \cap A_2 \mid B) = P(A_1)P(A_2)$
**f)** $P(A_1 \cap A_2) = P(A_1 \mid B)P(A_2 \mid B)$
**g)** $P(B \mid A_1 \cap A_2) = P(A_1)P(A_2)$

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In the next two questions, $Z$ is a $N(0,1)$ (standard normal) random variable with **c**df $\Phi$ and **p**df $\phi$, and $U$ is a Uniform$(0,1)$ random variable whose **p**df we denote by $h$.

*These questions are based on the two facts stated on page 17 and page 19 of notes3.pdf. See also page 21 of notes3.pdf, for an application of these facts to the Cauchy distribution.*

**Problem 9.** (3%) Which one of the following random variables has a Uniform$(0,1)$ distribution?

**a)** $\Phi^{-1}(U)$ **b)** $\Phi(U)$ **c)** $h(U)$ **d)** $\phi(U)$ **e)** $\Phi^{-1}(Z)$ **f)**$\star$ $\Phi(Z)$ **g)** $h(Z)$ **h)** $\phi(Z)$

**Problem 10.** (3%) Which one of the following random variables has a $N(0,1)$ distribution?

**a)**$\star$ $\Phi^{-1}(U)$ **b)** $\Phi(U)$ **c)** $h(U)$ **d)** $\phi(U)$ **e)** $\Phi^{-1}(Z)$ **f)** $\Phi(Z)$ **g)** $h(Z)$ **h)** $\phi(Z)$

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**Problem 11.** (4%) In lecture we gave a formula for the density of $Y = g(X)$ where $g$ is a smooth non-monotonic function and $X$ has a density $f_X$. This formula is partly given below:

$$f_Y(y) = \sum_{x \in g^{-1}(\{y\})} \boxed{\phantom{xxxxxx}} \quad \text{for } y \in \mathcal{Y}$$

What goes inside the box? Put your answer below.

*See page 24 of notes3.pdf. The complete formula is:*

$$f_Y(y) = \sum_{x \in g^{-1}(\{y\})} f_X(x) \left| \frac{1}{g'(x)} \right| \quad \textit{for } y \in \mathcal{Y}$$

*So what goes inside the box is* $f_X(x)\left|\frac{1}{g'(x)}\right|$. *There is another formula for* $f_Y(y)$ *which involves terms like* $f_X(g_i^{-1}(y))|\frac{d}{dy}g_i^{-1}(y)|$, *but this is the wrong response to this problem because it does not involve summing over* $x \in g^{(-1)}(\{y\})$.