TEST #2
STA 5326 **Name:**____________________________________
October 31, 2013

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all problems unless you are explicitly told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely, unless stated otherwise. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **10** pages.
- There are a total of **100** points.

Problem 1. (13%) For $0 < p < 1$ and any integer $n \geq 3$, show that

$$\sum_{k=0}^{2} \binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)(n-2)}{2} \int_0^{1-p} t^{n-3}(1-t)^2 \, dt \, .$$

This is a special case of exercise 2.40. The problem can be done by applying repeated integration by parts to the integral; only two integrations by parts are needed in this case. Another approach is to think of the left and right hand sides of the equation as functions of p*, call them* $L(p)$ *and* $R(p)$*. To show that* $L(p) = R(p)$ *for all* p *(with* $0 \leq p \leq 1$*), it suffices to show that* $L'(p) = R'(p)$ *for all* p *and that* $L(p_o) = R(p_0)$ *for some value* p_0*. The easiest value to use is* $p_0 = 1$ *for which it is clear that* $L(1) = R(1) = 0$*.*

The general result in exercise 2.40 gives an interesting connection between the Binomial and Beta distributions. There is a nice probability story for this result (which we are not likely to have the time to cover this year).

[**Problem 1 continued**] (Additional work space)

Problem 2. Consider the Pareto pdf defined by

$$f(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}\,, \qquad \alpha < x < \infty, \;\; \alpha > 0, \;\; \beta > 0\,.$$

This is exercise 3.23. The mean and variance can be read from the appendix. So there is no credit for getting the correct answer; all the credit is for the explanation and the work.

(a) (6%) Verify that $f(x)$ is a pdf.

(b) (6%) Compute the **mean** of this distribution, and specify the values of β for which the mean exists.

[**Problem 2 continued**]

(c) (6%) Compute the **variance** of this distribution, and specify the values of β for which the variance exists.

Problem 3.

(a) (6%) For a nonnegative random variable T, a formula expressing the hazard function $h(t)$ in terms of the pdf f and cdf F was stated in lecture and proved in the homework. What is this formula? (Just give the formula. No work is required.)

See exercise 3.25 and page 4 of notes7.pdf. The desired formula is $h(t) = \dfrac{f(t)}{1-F(t)}$*, but* $h(t) = -\dfrac{d}{dt}\log(1-F(t))$ *is also acceptable and should receive full credit.*

(b) (6%) If T has the Pareto pdf used in the previous problem

$$f(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}, \qquad \alpha < x < \infty, \ \ \alpha > 0, \ \ \beta > 0,$$

what is the hazard function of T?

$1 - F(t) = \dfrac{\alpha^\beta}{t^\beta}$ *for* $t \geq \alpha$ *so that* $h(t) = \dfrac{f(t)}{1-F(t)} = \dfrac{\beta}{t}$ *for* $t \geq \alpha$*. The hazard function is* $h(t) = 0$ *for* $t < \alpha$*.*

(c) (6%) Suppose a creature has lifetime T having the pdf given above with $\alpha = 2$ and $\beta = 3$. If the creature is alive at time 4.000, what is the approximate probability it will die before time 4.001?

Use the fact on page 4 of notes7.pdf: the probability that a creature which is alive at time t *will die before time* $t + \delta$ *is approximately* $h(t)\delta$ *for small values of* δ*.*

Problem 4. Suppose $X_1, X_2, \ldots, X_n$ are iid with the Pareto pdf used in the previous problems:

$$f(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}}\,, \qquad \alpha < x < \infty, \;\; \alpha > 0, \;\; \beta > 0\,.$$

Define $Z = \min X_i$.

(a) (8%) Find the cdf of Z.

Similar to C3. Use the result on page 7 of notes7.pdf. This leads to

$$F_Z(x) = 1 - (1 - F(x))^n = 1 - \left(\frac{\alpha^\beta}{x^\beta}\right)^n \quad \textit{for } x > \alpha\,.$$

Students can use the cdf F they found in Problem 3(b). If they made a mistake finding the cdf in 3(b) and use the same cdf here (but their solution is otherwise correct), give them all of the credit to avoid penalizing them twice for the same error.

(b) (7%) Find EZ

If you differentiate the cdf $F_Z(x) = 1 - \alpha^{n\beta}/x^{n\beta}$ for $x > \alpha$, you find the density of Z is

$$f_Z(x) = \frac{n\beta\,\alpha^{n\beta}}{x^{n\beta+1}} \quad \textit{for } x > \alpha\,,$$

which is the Pareto density with "β" set equal to $n\beta$. Thus, the formula for the mean given in Problem 2(b) and in the Appendix can also be used to find the mean of Z just by replacing β by $n\beta$. If students observe this, they con't need to recalculate the mean.

Problem 5. (15%) Suppose $X \sim \text{Beta}(\alpha = 200, \beta = 600)$. Calculate a normal approximation for $P(.254 < X < .280)$. (Your final answer should be a numerical value.)

Students should show work as if they were using the tables, but can use their calculators to actually compute the values. The Beta distribution is continuous, so nobody should use any sort of continuity correction here.

Using the formulas in the Appendix, the mean and standard deviation of the Beta distribution are

$$\mu = 0.25 \quad \textit{and} \quad \sigma = 0.01529975\,.$$

Using these values to compute z-scores for .254 and .280, the normal approximation becomes

$$P(0.2614422 < Z < 1.9608162) \quad \textit{where } Z \sim N(0,1)$$

The answer is now obtained by taking the difference between two values read from the normal table. A student actually using the tables should get about 0.3724, and someone using their calculator should get about 0.3719255. For reference, the exact value (computed using the R function **`pbeta`***) is 0.3649558.*

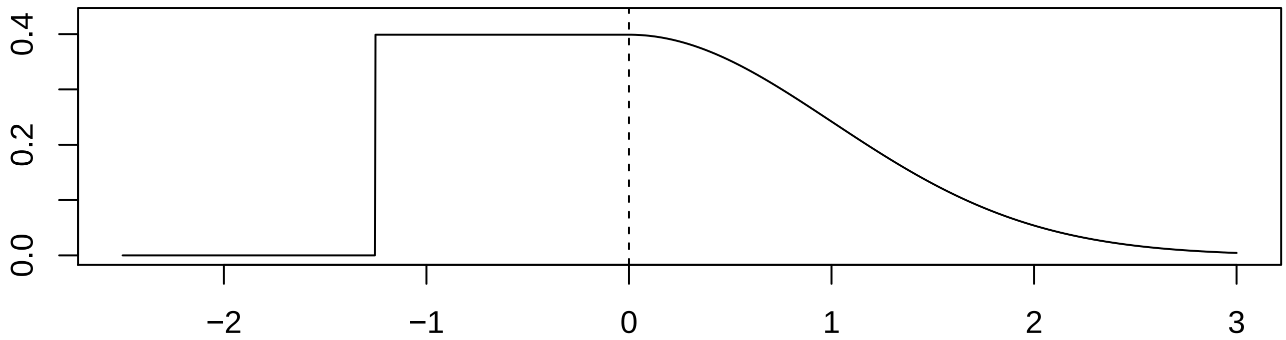
Problem 6. (12%) Show that if $X \sim \text{Geometric}(p)$, then

$$P(X > y + z \,|\, X > y) = P(X > z) \quad \text{for all integers } y, z > 0.$$

See pages 10 and 11 of notes6.pdf.

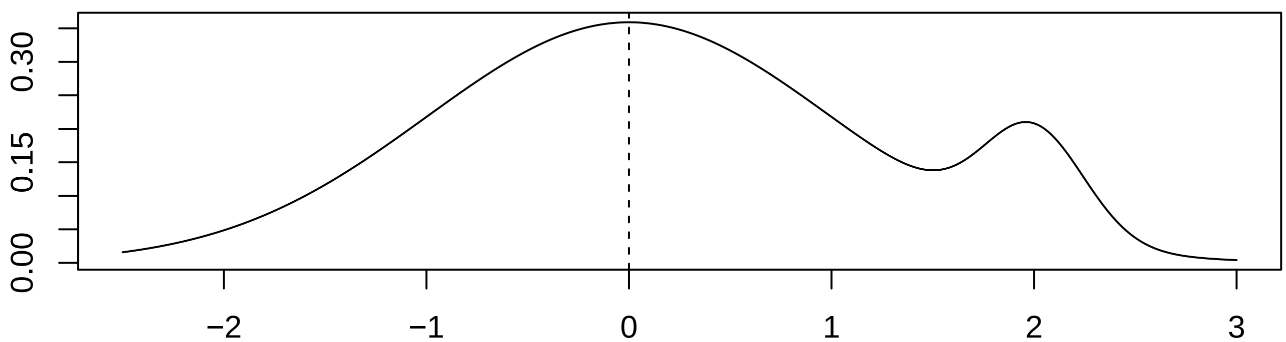
The term "unimodal" is sometimes used loosely, but a careful definition is given by the following: A pdf is said to be "unimodal with mode a" if $f(a) \ge f(x) \ge f(y)$ whenever $a \le x \le y$ or $a \ge x \ge y$. Use this definition to answer the following two questions. (Circle the correct response.) Assume that all pdf's given below are zero outside of the range pictured.

This is partly based on exercise 2.27. The correct answers are indicated by stars (⋆).



Problem 7. (3%) The pdf pictured **above** is unimodal with mode 0.

a)⋆ True **b**) False **c**) Cannot be determined from the given information



Problem 8. (3%) The pdf pictured **above** is unimodal with mode 0.

a) True **b**)⋆ False **c**) Cannot be determined from the given information

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Problem 9. (3%) Suppose X is a random variable with mean μ. The formula for the standardized **kurtosis** of X contains which one of the following quantities.

a) Ee^{tX} **b**) $\Gamma(X)$ **c**) $\Gamma(\alpha)$ **d**) $\dfrac{\Gamma(\alpha+3)}{\Gamma(\alpha)}$ **e**) $M_X^{(3)}(0)$

f) $\dbinom{n}{X}$ **g**) $f_X(x)$ **h**) $F_X(x)$ **i**)⋆ $E(X-\mu)^4$ **j**) $E(X-\mu)^3$

See page 15 of notes4.pdf.