TEST #3
STA 5326 **Name:**______________________________________
December 5, 2013

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.
- Show and explain your work (including your calculations) for all problems unless you are explicitly told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely, unless stated otherwise. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **12** pages.
- There are a total of **100** points.

Problem 1. (9%) Let X and Y be independent Geometric(p) random variables. We know from lecture that $X+Y$ has the Negative Binomial($r=2,p$) distribution (listed in the appendix as the "**alternate form**" of the Negative Binomial pmf). Find the distribution of $X \mid X+Y$. That is, for any integers j and k with $k \geq 2$ and $1 \leq j \leq k-1$, find the value of $P(X=j \mid X+Y=k)$.

Problem 2. Suppose that X and Y are independent with the same Cauchy density:

$$f_X(z) = f_Y(z) = \frac{1}{\pi(1+z^2)}, \quad -\infty < z < \infty.$$

Define $U = 2X + 3Y$ and $V = 2X - 3Y$.

(a) (15%) Find the joint density of (U, V).

(b) (4%) Are U and V independent? Answer "Yes" or "No" and justify your answer.

Problem 3. Suppose $X \mid P \sim \text{Binomial}(n, P)$ and $P \sim \text{Beta}(\alpha, \beta)$.

(a) (5%) Find EX.

(b) (5%) Find $\text{Var}(X)$.

[**Problem 3 continued**]

(c) (6%) Find the marginal distribution of X.

Problem 4. A pdf is defined by

$$f(x,y) = \begin{cases} C(x-y) & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

[Note: This pdf is positive on a triangle with corner points $(0,0)$, $(1,0)$, and $(1,1)$.]

(a) (6%) Find the value of C.

[**Problem 4 continued**]

(b) (5%) Find the marginal density of X.

(Note: If you could not find the value of C, just leave it as C in your answer to this and the following parts.)

[**Problem 4 continued**]

(c) (5%) Find $f_{Y|X}(y|x)$ for $0 < x < 1$.

(Make sure to specify the support of $f_{Y|X}(y|x)$, i.e., the values of y for which it is positive.)

(d) (5%) Find $E(Y \mid X = x)$ for $0 < x < 1$.

Problem 5. The table below gives the values of X, the pmf of X, and $E(Y \mid X)$ and $\mathrm{Var}(Y \mid X)$ expressed as functions of X.

X	$f_X(x)$	$E(Y\|X)$	$\mathrm{Var}(Y\|X)$
1	1/6	1	1/7
2	1/3	1/2	2/7
3	1/2	1/3	4/7

Using this table, do the following.

(a) (5%) Compute $E(Y)$.

(b) (5%) Compute $\mathrm{Var}[E(Y|X)]$.

Problem 6. A Markov chain has initial distribution $\boldsymbol{a}$ and transition probability matrix P given by

$$\mathrm{a} = (1/21, 2/21, 3/21, 4/21, 5/21, 6/21) \qquad P = \begin{pmatrix} 0 & .7 & 0 & 0 & 0 & .3 \\ .2 & 0 & .8 & 0 & 0 & 0 \\ 0 & .2 & 0 & .3 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ .6 & 0 & 0 & 0 & .4 & 0 \end{pmatrix}$$

Answer the following.

(a) (5%) Find $P(X_0 = 1, X_1 = 2, X_2 = 3, X_3 = 6, X_4 = 1)$.

(b) (5%) Which states communicate with state 4? Give your answer by filling in the table below with $\boldsymbol{Y}$ or $\boldsymbol{N}$ where $\boldsymbol{Y}$ means "Yes, does communicate" and $\boldsymbol{N}$ means "No, does **not** communicate."

State	1	2	3	4	5	6
Communicates with 4?						

[**Problem 6 continued**]

Recall that:

$$\mathrm{a} = (1/21, 2/21, 3/21, 4/21, 5/21, 6/21) \qquad P = \begin{pmatrix} 0 & .7 & 0 & 0 & 0 & .3 \\ .2 & 0 & .8 & 0 & 0 & 0 \\ 0 & .2 & 0 & .3 & 0 & .5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ .6 & 0 & 0 & 0 & .4 & 0 \end{pmatrix}$$

(c) (5%) The stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6)$ satisfies a system of linear equations. Write down this system of equations, but do NOT solve them. (Be explicit. Do **NOT** use matrix notation in your answer.)

Problem 7. (6%) Suppose random variables X and Y have a joint cdf

$$F(x, y) = \frac{x^3 y}{4} + \frac{3xy}{4} \quad \text{for } 0 < x < 1, 0 < y < 1.$$

Let A be a very small square with sides of length $.001 = 10^{-3}$ which contains the point $(1/2, 1/2)$. Compute a simple (but good) numerical approximation to $P((X, Y) \in A)$.

The last question is multiple choice. Circle the correct response. No work is required.

Problem 8. (4%) Suppose a_1, a_2, c, b_1, b_2, d are constants, and X_1, X_2, Y_1, Y_2 are random variables with finite variances. Then

$$\text{Cov}(a_1X_1 + a_2X_2 + c, b_1Y_1 + b_2Y_2 + d) = \ldots$$

a) $a_1b_1\text{Var}(X_1) + 2a_1b_2\text{Cov}(X_1, Y_2) + 2a_2b_1\text{Cov}(X_2, Y_1) + a_2b_2\text{Var}(X_2)$

b) $a_1b_1\text{Var}(X_1) + 2a_1b_2\text{Cov}(X_1, Y_2) + 2a_2b_1\text{Cov}(X_2, Y_1) + a_2b_2\text{Var}(Y_2)$

c) $a_1b_1\text{Var}(X_1) + 2a_1b_2\text{Cov}(X_1, Y_2) + 2a_2b_1\text{Cov}(X_2, Y_1) + a_2b_2\text{Var}(X_2) + cd$

d) $a_1^2\text{Var}(X_1) + 2a_1a_2\text{Cov}(X_1, X_2) + a_2^2\text{Var}(X_2) + b_1^2\text{Var}(Y_1) + 2b_1b_2\text{Cov}(Y_1, Y_2) + b_2^2\text{Var}(Y_2)$

e) $a_1^2\text{Var}(X_1) + 2a_1a_2\text{Cov}(X_1, X_2) + a_2^2\text{Var}(X_2) + b_1^2\text{Var}(Y_1) + 2b_1b_2\text{Cov}(Y_1, Y_2) + b_2^2\text{Var}(Y_2) + cd$

f) $a_1b_1\text{Cov}(X_1, Y_1) + a_1b_2\text{Cov}(X_1, Y_2) + a_2b_1\text{Cov}(X_2, Y_1) + a_2b_2\text{Cov}(X_2, Y_2) + cd$

g) $a_1b_1\text{Cov}(X_1, Y_1) + a_1b_2\text{Cov}(X_1, Y_2) + a_2b_1\text{Cov}(X_2, Y_1) + a_2b_2\text{Cov}(X_2, Y_2)$