TEST #1
STA 5326 **Name:**__________________________________
September 25, 2014

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- The different problems are not related. The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- Show and explain your work (including your calculations) for all problems unless you are explicitly told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- The grader should be able to see how you got from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **12** pages.
- There are a total of **100** points.

Problem 1. Let α, β be two fixed positive constants, and define the density (pdf)

$$f(x) = \begin{cases} \dfrac{\alpha}{2} e^{\alpha x} & \text{if } x < 0\,, \\ \dfrac{\beta}{2} e^{-\beta x} & \text{if } x \geq 0\,. \end{cases}$$

Suppose X is a random variable with density f.

The parts of this problem are similar to exercise 2.4.

(a) (8%) Find the cumulative distribution function (cdf) of X.

The answer is

$$F_X(x) = \begin{cases} \frac{1}{2} e^{\alpha x} & \textit{if } x < 0\,, \\ \frac{1}{2} + \frac{1}{2}(1 - e^{-\beta x}) & \textit{if } x \geq 0\,. \end{cases}$$

[**Problem 1 continued**]
Recall that:

$$f(x) = \begin{cases} \dfrac{\alpha}{2} e^{\alpha x} & \text{if } x < 0\,, \\ \dfrac{\beta}{2} e^{-\beta x} & \text{if } x \geq 0\,. \end{cases}$$

(b) (6%) Find $P(|X| < t)$ for all $t > 0$.

The answer is

$$P(|X| < t) = 1 - \frac{1}{2}(e^{-\alpha t} + e^{-\beta t}) \quad \textit{for } t > 0\,.$$

Problem 2. (10%) An urn contains 16 balls. There are 4 red balls, 4 green balls, 4 blue balls, and 4 rainbow balls. Five balls are drawn at random with**OUT** replacement. What is the probability that all five balls are the same color if rainbow balls are allowed to match any other color? (Rainbow balls are "wild.")

See notes1, pages 17-18: the "five of a kind in poker" example.
Let $A = \{$*all five balls are the same color*$\}$ *and* A_i *be the event that "five of a kind" is achieved by drawing* i *rainbow balls and* $5 - i$ *balls of some other color. By the same argument in the notes we get*

$$\begin{aligned} P(A) &= P(A_1) + P(A_2) + P(A_3) + P(A_4) \\ &= \frac{\sum_{i=1}^{4} \binom{4}{i} \cdot 3 \cdot \binom{4}{5-i}}{\binom{16}{5}} \\ &= \frac{168}{4368} = 0.03846154 \end{aligned}$$

Problem 3. Consider the function:

$$F(x) = \begin{cases} 2e^x & \text{for } x < -\ln 3 \\ 2/3 & \text{for } -\ln 3 \le x < \ln 6 \\ 1 - 2e^{-x} & \text{for } x \ge \ln 6 \end{cases}$$

Parts (a) and (c) are similar to exercise 2.8(b).

(a) (3%) State the properties of a cumulative distribution function (cdf).

See notes2, page 22.

(b) (7%) Show that F is a cdf.

This is similar to part of exercise 2.8.

[Problem 3 continued]
Recall that:

$$F(x) = \begin{cases} 2e^x & \text{for } x < -\ln 3 \\ 2/3 & \text{for } -\ln 3 \le x < \ln 6 \\ 1 - 2e^{-x} & \text{for } x \ge \ln 6 \end{cases}$$

(c) (6%) Find $F^{-1}(y)$ for $0 < y < 1$.
(Note: If F has any flat spots, we define $F^{-1}(y) = \inf\{x : F(x) \ge y\}$.)

This is similar to part of exercise 2.8.
The answer is:

$$F^{-1}(y) = \begin{cases} \ln(y/2) & \textit{for } 0 < y < 2/3 \\ -\ln 3 & \textit{for } y = 2/3 \\ -\ln((1-y)/2) & \textit{for } 2/3 < y < 1 \end{cases} = \begin{cases} \ln(y/2) & \textit{for } 0 < y \le 2/3 \\ -\ln((1-y)/2) & \textit{for } 2/3 < y < 1 \end{cases}$$

Problem 4. Three players, A, B, and C, play a game in which they take turns shooting at a target. They shoot in the order A, B, C, A, B, C, A, B, C, ... Assume all shots are independent with probability $\boldsymbol{p}$ of hitting the target. The players continue shooting until someone hits the target. Then the game stops (immediately) and that person is declared the winner.

(a) (8%) What is the probability that player $\boldsymbol{B}$ is the winner?

This is similar to exercise 1.24.

Let D_i be the event that the i-th shot is the first to hit the target. Then

$$\begin{aligned} P(B \text{ wins}) = P(D_2) + P(D_5) + P(D_8) + \ldots \\ &= (1-p)p + (1-p)^4 p + (1-p)^7 p + \ldots \\ &= \frac{(1-p)p}{1-(1-p)^3} \end{aligned}$$

(b) (6%) What is the probability that more than 10 shots are fired during the entire game?

This is similar to exercise 1.26.
Answer: $P(\text{more than 10 shots fired}) = P(\text{first 10 shots all miss}) = (1-p)^{10}$

Problem 5. (14%) A jar contains 50 balls. Each ball is labeled with a letter and a digit. The 50 balls in the jar are labeled as follows:

$A0$	$A1$	$A2$	$A3$	$A4$	$A5$	$A6$	$A7$	$A8$	$A9$
$B0$	$B1$	$B2$	$B3$	$B4$	$B5$	$B6$	$B7$	$B8$	$B9$
$C0$	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$	$C7$	$C8$	$C9$
$D0$	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$	$D7$	$D8$	$D9$
$E0$	$E1$	$E2$	$E3$	$E4$	$E5$	$E6$	$E7$	$E8$	$E9$

In a simple game, a player draws out **four** balls at random with**OUT** replacement. A player wins cash prizes if any of the following occur:

- All four letters are the same.
- There are repeated digits. (Two or more of the digits take the same value.)
- The digits can be placed in a sequence.
  (There are 7 possible sequences: 0123, 1234, ..., 5678, 6789.)

What is the probability that a person playing this game will win **nothing** at all?

This is similar to exercise B5.
The solution exactly parallels that of B5, and the answer is:

$$\begin{aligned} P(A \cap B^c \cap C^c) &= P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) \\ &= \frac{\binom{10}{4} \cdot 5^4 - 7 \cdot 5^4 - \binom{10}{4} \cdot 5 + 7 \cdot 5}{\binom{50}{4}} \\ &= 0.5699088146 - 0.0189969605 - 0.0045592705 + 0.0001519757 \\ &= 0.5465045593 \end{aligned}$$

Here

$$\begin{aligned} A &= \{ \textit{ no repeated digits } \}, \\ B &= \{ \textit{ digits can be placed in a sequence } \}, \\ C &= \{ \textit{ all four letters are the same } \}. \end{aligned}$$

Numerical answers are not required.

[**Problem 5 continued**]
Extra work space if you need it.

Problem 6. (10%) Suppose $Y = 3X + 5$ and $f_X(x) = 30x^2(1-x)^2$, $0 < x < 1$. Find the density of Y.

This is similar to the parts of exercise 2.1.

Problem 7. A hat contains two coins. When tossed, Coin #1 has probability **1/5** of landing heads, and Coin #2 has probability **4/5** of heads. A coin is chosen at random from the hat, and this same coin is tossed repeatedly. Let A_i be the event that the i-th toss is heads.

(a) (10%) Calculate $P(A_2 \,|\, A_1) = P(\text{second toss is heads} \,|\, \text{first toss is heads})$.

See notes2, pages 7-8.

(b) (2%) What is the approximate value of $P(A_{100} \,|\, A_1 \cap A_2 \cap \cdots \cap A_{98} \cap A_{99})$?
(Just state the answer. No work is required.)

Answer: 4/5

Problem 8. (7%) Suppose $X \sim \text{Normal}(0,1)$ and $Y = g(X)$ where $g(x) = (x-1)(x-2)(x-3)$. What is the value of $f_Y(0)$, the density of Y at the value 0? **State the formula you are using!**

(Hint: consider $g^{-1}(\{0\})$. You may write the normal density as $\phi(x)$.)

Use the formula on notes3, page 24.

Problem 9. (3%) State a formula for $P(A$ **or** B **but not both**$)$ in terms of the quantities $P(A)$, $P(B)$, and $P(A \cap B)$.
(Just state the answer. No work is required.)

Answer: $P(A) + P(B) - 2P(A \cap B)$.