TEST #2
STA 5326 **Name:**______________________________________
October 30, 2014

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**. (You will have access to a copy of the "Table of Common Distributions" given in the back of the text.)
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- The different problems are not related. The different parts of a problem are sometimes unrelated. If you cannot solve part of a problem, you should still go on to look at the later parts.
- Show and explain your work (including your calculations) for all problems unless you are explicitly told otherwise. **No credit is given without work.** But don't get carried away! Show enough work so that what you have done is clearly understandable.
- The grader should be able to see how you got from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- If you need more space for a problem than what is given, use the back of the same page and write **Work on Back** to indicate you have done so.
- On problems where work is shown, **circle your answer**. (On these problems, you should give only one answer!)
- All the work on the exam should be your own. No "cooperation" is allowed.
- Arithmetic does **not** have to be done completely **unless requested otherwise**. Answers can be left as fractions or products. You do not have to evaluate binomial coefficients, factorials or large powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- You need only pens, pencils, erasers and a calculator. (You will be supplied with scratch paper.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **12** pages.
- There are a total of **100** points.

Problem 1. An urn contains R red balls and G green balls. A player randomly selects **10** balls, doing this one by one and with**OUT** replacement. (Assume $R+G \geq 10$.) The player wins a dollar every time he selects three red balls in a row. To be more precise, a player receives a dollar after the i-th draw, if he selected red balls on draws $i-2$, $i-1$, and i. Note that under these rules, a player drawing 4 red balls in a row receives a total of 2 dollars; a player drawing 5 red balls in a row receives a total of 3 dollars, etc. Let X be the player's total winnings.

(a) (9%) Find EX.

[**Problem 1 continued**]

(b) (9%) Find EX^2.

Problem 2.

(a) (9%) Prove the "two-way" rule for expectation (also known as the "law of the unconscious statistician") which says that $Eg(X) = EY$ where $Y = g(X)$. Assume that $g(x)$ is a strictly increasing differentiable function and that X has a density.

[**Problem 2 continued**]

(b) (8%) Consider a random right triangle with vertices $(0,0)$, $(b,0)$, and (b,Y) where $b > 0$ is a constant and Y is random. This triangle is constructed so that the angle at the origin (call it X) is a random variable with density

$$f_X(x) = \cos(x) \quad \text{for } 0 < x < \pi/2.$$

Here Y is the height of the random triangle determined by the angle X. Use the "two-way" rule to find EY.

Problem 3. (16%) Al and Bob each roll a fair six-sided die 100,000 times and count the number of 1's they observe. Use a normal approximation to find the probability they both roll exactly the same number of 1's.

Show the argument and do NOT just quote a result from lecture. For full credit, your answer must be a **numerical value** and you must obtain your answer **withOUT using normal tables or your calculator's Φ function**!

[Problem 3 continued]

[Additional work space if needed]

Problem 4. Let X have a discrete uniform distribution on N_0 to N_1 that puts equal probability on each of the integers $N_0, N_0 + 1, \ldots, N_1 - 1, N_1$.

(a) (8%) Find EX

[**Problem 4 continued**]

(b) (8%) Find $\mathrm{Var}(X)$.

Problem 5. (16%) Suppose the random variables Y and $X_1, X_2, X_3, \ldots$ satisfy

$$Y \sim \text{Exponential}(1) \quad \text{and}$$
$$X_n \sim \text{Geometric}\left(\frac{1}{n}\right) \quad \text{for all } n.$$

Use moment generating functions (mgf's) to prove that

$$\frac{X_n}{n} \xrightarrow{d} Y \quad (\text{as } n \to \infty).$$

[Problem 5 continued]

[Additional work space if needed]

Problem 6.

(a) (4%) Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. with cumulative distribution function (cdf) F. State a formula for the **cdf** of $Y = \max\limits_{1\le i\le n} X_i$. (No work is required.)

(b) (3%) Now take $n = 3$ and assume X_1, X_2, X_3 are i.i.d. Exponential(β) random variables. Apply the result of part (a) to find the **cdf** of $Y = \max\limits_{1\le i\le 3} X_i$. (No work is required.)

(c) (10%) Use the result of part (b) to find EY. **Show your work.** (Do NOT use the heuristic argument based on the memoryless property. If you wish, you may use a certain result from homework without proof.)