TEST #2
STA 5326 **Name:**________________________________
October 29, 2015

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**.
- You will have access to a copy of the "Table of Common Distributions" given in the back of the text and to a table of the normal distribution.
- For problems requiring probability approximations or the use of tables, you must give a numerical answer in order to receive full credit.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- Show and explain your work (including your calculations) for all problems except those on the last two pages. **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- All the work on the exam should be your own. No "cooperation" is allowed.
- You need to bring only what you write with: pens, pencils, erasers, ... (You will be supplied with scratch paper.) A calculator would be somewhat helpful, but is not absolutely essential. (The arithmetic is not too bad.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** pages.
- There are a total of **100** points.

Problem 1. (16%) Suppose λ and c are arbitrary real numbers. Let $p_1, p_2, p_3, \ldots$ be any sequence of values such that $p_n \to 1$ and $n(1-p_n) \to \lambda$ as $n \to \infty$. Evaluate the following limit. Show your work.

$$\lim_{n\to\infty} \left(\frac{p_n}{1-(1-p_n)c}\right)^n$$

The answer is $e^{\lambda(c-1)}$. See Exercise 3.15. The calculation of this limit is essentially the solution of Exercise 3.15.

Solution:

$$\frac{p_n}{1-(1-p_n)c} = 1 + \frac{(1-p_n)(c-1)}{1-(1-p_n)c} = 1 + \frac{1}{n} \cdot \frac{n(1-p_n)(c-1)}{1-(1-p_n)c}.$$

Since

$$\lim_{n\to\infty} \frac{n(1-p_n)(c-1)}{1-(1-p_n)c} = \lambda(c-1),$$

we conclude that

$$\lim_{n\to\infty} \left(\frac{p_n}{1-(1-p_n)c}\right)^n = \left(1 + \frac{1}{n} \cdot \frac{n(1-p_n)(c-1)}{1-(1-p_n)c}\right)^n = e^{\lambda(c-1)}.$$

Problem 2. (16%) Suppose X has density $f_X(x) = \frac{1}{\beta}e^{-x/\beta}$ for $x \geq 0$. Find the variance of $Y = X^\delta$ for $\delta > 0$.
(Show your work. **Do NOT use results from the appendix** in your solution to this problem.)

This is exercise 3.24(a).

I do not allow the students to quote results from the appendix on this problem because the section on the Exponential distribution mentions that $X^{1/\gamma}$ has a Weibull distribution for any $\gamma > 0$, and the section on the Weibull gives the variance of the Weibull distribution. So if one uses the appendix, one can answer this question without doing any work whatsoever.

Problem 3. A standard drug is known to be effective in 75% of the cases in which it is used. A less expensive new drug is currently being tested.

(a) (14%) The new drug is tested on 432 patients and found to be effective in only 301 cases. Is the new drug **inferior** to the old one? (Answer this question by computing an approximation to a relevant probability and then drawing a conclusion from this.)

The problem is similar to 3.5.

To receive full credit, a student must produce a numerical final answer.

The solution requires a normal approximation with a continuity correction. If a student neglects the continuity correction but their solution is otherwise correct, they should lose (maybe) 2 points. If they do the continuity correction in the wrong direction, then the answer is even worse than if they did not do it at all, so they should still lose (maybe) 2 points.

Solution: Let X = # *of patients on which new drug is effective.*

$X \sim \textit{Binomial}(n = 432, p = .75) \approx N(\mu = 432 * .75 = 324, \sigma^2 = 432 * .75 * .25 = 81)$

$\sigma = \sqrt{81} = 9$

Let $X^* \sim N(324, 81)$ *and* $Z \sim N(0, 1)$. *Using the continuity correction, we get*

$$\begin{aligned} P(X \le 301) &\approx P(X^* \le 301.5) = P(Z \le (301.5 - 324)/9) \\ &= P(Z \le -2.5) = \Phi(-2.5) = 1 - \Phi(2.5) = 1 - .99379 = .00621 \end{aligned}$$

If one neglects the continuity correction, the answer is 0.00530.

If the two drugs are equally effective, the probability of getting 301 or fewer cases is quite small, so we are inclined to believe the new drug is inferior.

Note: The exact binomial probability is:
pbinom(301,432,.75)
0.007014883

[**Problem 3 continued**]

(b) (12%) Suppose now that each drug is tested on 600 patients (so that there are 1200 patients in total). If the two drugs are actually equally effective, what is the approximate probability that the old drug will be effective on **at least 23 more** patients than the new drug?

This problem is similar to C5. It also resembles normal approximation to Binomial problems like 3.5.

This part also requires a numerical answer to receive full credit.

The bulk of the credit for this part should be on the normal approximation to $Y - X$ (since the previous part already tested students on the fact that X and similarly Y are Binomial rv's and are approximately normal).

Students should lose (maybe) 2 points if they don't do the continuity correction or do it in the wrong direction.

Solution: Let X = # of patients on which new drug is effective, and Y = # for which old drug is effective.

Now X and Y are independent Binomial(n=600,p=.75) which is approximately Normal with mean $600 * .75 = 450$ *and variance* $600 * .75 * .25 = 112.5$. *Therefore, by results from lecture, $Y - X$ is approximately normal with mean 450-450 = 0 and variance 2*112.5 = 225 with standard deviation* $\sqrt{225} = 15$.

Let $D = Y - X$. We desire $P(D \geq 23)$. Using the continuity correction,

$$P(D \geq 23) \approx P(D^* \geq 22.5) = P(Z \geq (22.5 - 0)/15)$$
$$= P(Z \geq 1.5) = 1 - \Phi(1.5) = 0.0668072$$

Problem 4. (15%) A carnival game has a bucket of n hollow balls, each containing a prize. One of the prizes is an expensive ring; the others are cheap trinkets. If you pay a dollar, you get to draw a ball from the bucket and keep the prize. A man decides to keep playing (and keep paying a dollar per ball) until he wins the ring. **The bucket is only refilled with balls after each player is completely finished playing.** Let X be the amount of money the man ends up paying in his quest for the ring. Find EX.

This is problem 3.4(b) with a different story.

Solution: X has a discrete uniform distribution on the integers 1, 2, ...,n. From the appendix, this distribution has mean $(n+1)/2$. So $EX = (n+1)/2$. This solution is short, but correct, and should receive full credit.

The students don't have to prove that X is discrete uniform since it is fairly intuitive and similar to facts in lecture. Suppose the balls are labeled 1, 2, ..., n and ball #1 contains the ring. If the balls are drawn one by one at random from the bucket until it is empty, all $n!$ possible orderings of the balls are equally likely, and ball #1 is equally likely to be in any of the n possible positions. Hence X has a discrete uniform distribution.

If the balls were being drawn with replacement, then X would have a geometric distribution with $p = 1/n$ which has a mean of $1/p = n$. If students give this solution, they should get roughly half credit (but no more, since I told the class the balls were NOT being replaced).

The remaining problems require no work. You will receive full credit just for stating the correct answer. However, some work space is provided in case needed. In the multiple choice problems, circle the single correct response **except in one problem** which is clearly identified.

Most of the remaining problems are worth 3%, but Problem #5 is worth 5% and Problem #9 is worth 4%. Partial credit may be given on Problems #5, #6, and #9.

Problem 5. (5%) Let T_r denote the time of the r^{th} arrival in a Poisson process with constant rate λ. Express $P(T_r > t)$ as an explicit summation of simple terms. (Hint: Use a relationship between T_r and S_t, the number of arrivals by time t.)

See the discussion on pages 16-17 of notes7.pdf. The desired summation is given on page 16 of notes7.pdf.

Answer: $\displaystyle\sum_{i=0}^{r-1} \frac{(\lambda t)^i e^{-\lambda t}}{i!}$

Problem 6. (3%) Suppose the random variable X has density $f_X(x) = 5e^{-5x}$ for $x \geq 0$. What is the hazard function of X?

Answer: The hazard function is a constant function equal to 5, i.e., $h(t) = 5$ for all t.

The answer "$h(t) = 1/5$" or just "$h(t) =$ constant" should receive partial credit, maybe 1.5% or (if you are not giving half points) just 1%.

Note: *There are two versions of the exam which differ only in the ordering of the responses in the multiple choice questions.*

Problem 7. (3%) Suppose X is a random variable with density $f_X(x) = \frac{1}{2}e^{-|x|}$ for $-\infty < x < \infty$, which is a special case of the double exponential distribution. Let $M_X(t)$ denote the moment generating function (mgf) of X. What is the value of $M_X(2)$?

a) $-1/3$ **b)** $1/3$ **c)** -1 **d)** 1 **e)** $-\frac{e^2}{3}$

f) $\frac{e^2}{3}$ **g)** $-\frac{e}{3}$ **h)** $\frac{e}{3}$ **i)**$\star$ ∞ **j)** 0

Problem 8. (3%) Suppose X is a random variable with density $h(x)$ which satisfies $h(x) > 0$ for $x \geq 1$ and $h(x) = 0$ for $x < 1$. Let $\cdot$ denote multiplication. The integral given below

$$\int_0^\infty t \cdot h\left(e^{\sqrt{t}}\right) \cdot \left(\frac{d}{dt} e^{\sqrt{t}}\right) dt$$

is equal to ...

a) $E\left[\frac{\sqrt{t}}{2} e^{\sqrt{X}}\right]$ **b)**$\star$ $E\left[(\log X)^2\right]$ **c)** $E\left[e^{\sqrt{X}}\right]$ **d)** $E\left[h(e^{\sqrt{X}})\right]$

e) mgf of $\sqrt{X}$ **f)** mgf of $\log X$ **g)** mgf of $e^{\sqrt{X}}$ **h)** mgf of $\frac{\sqrt{t}}{2} e^{\sqrt{X}}$

In the next two problems, suppose $X_1, X_2, X_3, \ldots$ are iid with mean μ and variance $\sigma^2 < \infty$. Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$.

Problem 9. (4%) Circle **ALL** of the true statements below.

a)$\star$ $P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1$

b) $\lim_{n\to\infty} P\left(|\bar{X}_n - \mu| > \varepsilon\right) = 1$ for all $\varepsilon > 0$

c) $\lim_{n\to\infty} P\left(\bar{X}_n = \mu\right) = 1$

d)$\star$ $\lim_{n\to\infty} P\left(|\bar{X}_n - \mu| < \varepsilon\right) = 1$ for all $\varepsilon > 0$

e) $\lim_{n\to\infty} P\left(|\bar{X}_n - \mu| = 0\right) = 1$

f) $P\left(\lim_{n\to\infty} |\bar{X}_n - \mu| > \varepsilon\right) = 1$ for all $\varepsilon > 0$

In the above problem, give 2% for each correct response circled , and subtract 2% for each wrong response circled, but don't give negative scores. So if a student circles one right and one wrong answer, they get a zero. If a student circles two right and one wrong answer, they get 2%, etc.

Problem 10. (3%) Which ONE of the following statements is true? Circle the single correct response.

a) $P\left(|\bar{X}_n - \mu| > \varepsilon\right) > \frac{\sigma^2}{n\varepsilon^2}$ **b)**$\star$ $P\left(|\bar{X}_n - \mu| > \varepsilon\right) < \frac{\sigma^2}{n\varepsilon^2}$ **c)** $P\left(|S_n - n\mu| > \varepsilon\right) < \frac{\sigma^2}{n\varepsilon^2}$

d) $P\left(|\bar{X}_n - \mu| > \varepsilon\right) > \frac{n\sigma^2}{\varepsilon^2}$ **e)** $P\left(|S_n - n\mu| > \varepsilon\right) < \frac{n\varepsilon^2}{\sigma^2}$ **f)** $P\left(|S_n - n\mu| > \varepsilon\right) > \frac{n\varepsilon^2}{\sigma^2}$

Problem 11. (3%) Consider a distribution with density

$$f(x) = \frac{1}{10} e^{-|x-3|/5}, \quad -\infty < x < \infty.$$

What is the **median** of this distribution?

a)$\star$ 3 **b)** 5 **c)** 9 **d)** 10 **e)** 25 **f)** Does NOT exist

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Problem 12. (3%) Consider a distribution with density

$$f(x) = \frac{1}{2} \frac{1}{(1 + |x-3|)^2}, \quad -\infty < x < \infty.$$

What is the **mean** of this distribution?

a) 0 **b)** 2 **c)** 3 **d)** 4 **e)** 9 **f)**$\star$ Does NOT exist