TEST #3
STA 5326 **Name:**____________________________________
December 3, 2015

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- This exam is **closed book** and **closed notes**.
- You will have access to a copy of the "Table of Common Distributions" given in the back of the text.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- Show and explain your work (including your calculations) for all problems (except those on the last page). **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- All the work on the exam should be your own. No "cooperation" is allowed.
- You need to bring only what you write with: pens, pencils, erasers, . . . (You will be supplied with scratch paper.) A calculator is NOT needed for this exam.
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **9** pages.
- There are a total of **100** points.

Problem 1. (12%) Prove that if the joint cdf of X and Y satisfies

$$F_{X,Y}(x,y) = F_X(x)F_Y(y) \quad \text{for all } x, y,$$

then for any values a and b,

$$P(X > a, Y > b) = P(X > a)P(Y > b)\,.$$

Explain your argument.

This is a variation of Exercise 4.9.

The solution manual solution to 4.9 is not explained well (and actually is a little wrong). Give students full credit if their argument is as detailed as the solution manual.

A Correct Solution: For any two events A and B we know that:

$$\begin{aligned} P(B^c) &= 1 - P(B) \\ P(A \cap B^c) &= P(A) - P(A \cap B) \end{aligned}$$

Using these facts repeatedly we obtain:

$$\begin{aligned} P(X > a, Y > b) &= P(X > a) - P(X > a, Y \le b) \\ &= [1 - P(X \le a)] - [P(Y \le b) - P(X \le a, Y \le b)] \\ &= 1 - F_X(a) - F_Y(b) + F_{X,Y}(a,b) \\ &= 1 - F_X(a) - F_Y(b) + F_X(a)F_Y(b) \\ &= (1 - F_X(a))(1 - F_Y(b)) \\ &= P(X > a)P(Y > b) \end{aligned}$$

An Only Partly Correct Solution: It is true that

$$\begin{aligned} &F_{X,Y}(x,y) = F_X(x)F_Y(y) \quad \textit{for all } x, y, \\ &\implies \quad X \textit{ and } Y \textit{ are independent rv's} \\ &\implies \quad P(X > a, Y > b) = P(X > a)P(Y > b)\,. \end{aligned}$$

However, the fact used above (i.e., that the first line implies the second), although well known, was never stated in the lecture notes and cannot be used by the students unless they first prove it. Also, using this fact renders the problem trivial and defeats the purpose of the problem, which is to give a direct proof of a special case. Students who give the above proof **should only receive half credit**. *(They should have raised their hands and asked if this proof was allowed.)*

Problem 2. (14%) Let X, Y, Z be independent random variables with means μ_X, μ_Y, μ_Z and variances $\sigma_X^2, \sigma_Y^2, \sigma_Z^2$. Find an expression for the correlation between $XY+5$ and $YZ-6$ in terms of these means and variances.

This is similar to Exercise 4.42, but somewhat messier.

$$
\begin{aligned}
Corr(XY+5, YZ-6) &= \frac{Cov(XY+5, YZ-6)}{\sqrt{Var(XY+5)\cdot Var(YZ-6)}} = \frac{Cov(XY, YZ)}{\sqrt{Var(XY)\cdot Var(YZ)}} \\
&= \frac{E(XY^2Z) - (EXY)(EYZ)}{\sqrt{(EX^2Y^2 - (EXY)^2)\cdot(EY^2Z^2 - (EYZ)^2)}} \\
&= \frac{\mu_X(\sigma_Y^2 + \mu_Y^2)\mu_Z - (\mu_X\mu_Y)(\mu_Y\mu_Z)}{\sqrt{((\sigma_X^2+\mu_X^2)(\sigma_Y^2+\mu_Y^2) - \mu_X^2\mu_Y^2)\cdot((\sigma_Y^2+\mu_Y^2)(\sigma_Z^2+\mu_Z^2) - \mu_Y^2\mu_Z^2)}} \\
&= \frac{\mu_X\sigma_Y^2\mu_Z}{\sqrt{(\sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \mu_X^2\sigma_Y^2)\cdot(\sigma_Y^2\sigma_Z^2 + \sigma_Y^2\mu_Z^2 + \mu_Y^2\sigma_Z^2)}}
\end{aligned}
$$

This can be re-expressed in various way, but this is good enough. If students drop the constants $+5$ and -6 immediately, that is OK.

Problem 3. Suppose $X \mid P \sim \text{Negative Binomial}(r, P)$ and $P \sim \text{Beta}(\alpha, \beta)$.

This is Exercise 4.34(b). The solution manual has a reasonably good solution for this problem.

(a) (14%) Find EX.

[**Problem 3 continued**]

(b) (14%) Find the marginal pmf of X.

Problem 4. Suppose the joint pdf of (X, Y) is

$$f(x,y) = 24y(x-y) \quad \text{for } 0 < y < x < 1.$$

(The support of this joint pdf is a triangular region bounded by the lines $y = 0$, $x = 1$, and $y = x$.)

This situation is similar to the joint density with triangular support used in notes9.pdf and notes10.pdf.

(a) (10%) Calculate $F_{X,Y}(1/2, 2)$, the value of the joint cdf at the point $(1/2, 2)$.

Answer: $F_{X,Y}(1/2, 2) = 1/16$

Let D be the triangular region (call it D) which is the support of the joint density. $F_{X,Y}(1/2, 2)$ *is the probability of the corner-shaped region (call it A) with corner point* $(1/2, 2)$. *This probability may be found by integrating over* $A \cap D = \{(x, y) : 0 < y < x < 1/2\}$.

$$\begin{aligned}
F_{X,Y}(1/2,2) &= \iint_{A\cap D} 24y(x-y)\,dx\,dy = \int_0^{1/2}\int_0^x 24y(x-y)\,dy\,dx \\
&= \int_0^{1/2}\int_0^x 24(xy-y^2)\,dy\,dx = \int_0^{1/2} 24(xy^2/2 - y^3/3)\Big|_{y=0}^{y=x}\,dx \\
&= \int_0^{1/2} 4x^3\,dx = x^4\Big|_0^{1/2} = 1/16
\end{aligned}$$

[Problem 4 continued]

(b) (16%) Let

$$U = \frac{1}{X} \quad \text{and} \quad V = \frac{Y}{X}\,.$$

Find the joint pdf of (U, V). (Don't forget to specify the support of the joint pdf.)

This is similar to the example in notes10.pdf on pages 6-10. It is also similar in a general way to the bivariate transformation problems in the homework exercises.

Answer: $f_{U,V}(u, v) = 24 \cdot \dfrac{1}{u^5} \cdot v(1 - v)$ *for* $1 < u < \infty$, $0 < v < 1$.

The support is an important part of the answer (in this part and the next), and students should lose at least two points if they forget to give the support (particularly since I included reminders on the exam that they should do so).

Sketch of solution:

The transformation $U = 1/X$, $V = Y/X$ *maps the triangular region into the strip* $(1, \infty) \times (0, 1) = \{(u, v) : 1 < u < \infty, 0 < v < 1\}$.

The inverse transformation is $X = 1/U$, $Y = V/U$.

The Jacobian of the inverse transformation is $J = -1/u^3$ *so that* $|J| = 1/u^3$.

Thus, the answer is $f_{U,V}(u, v) = f_{X,Y}(1/u,\, v/u) \cdot 1/u^3$ *for* $1 < u < \infty\,, 0 < v < 1$ *which leads to the expression given above.*

[**Problem 4 continued**]

(c) (10%) Find the marginal pdf of V. (Don't forget to specify the support.)

See notes10.pdf, pages 9-11 for a similar example.

The answer is $f_V(v) = 6v(1-v)$ *for* $0 < v < 1$.

The marginal for V *may be obtained by integrating over* u *in the joint density:*

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du = \int_1^{\infty} \frac{24v(1-v)}{u^5}\,du = 24v(1-v)\int_1^{\infty} u^{-5}\,du = 6v(1-v)\,,\ 0 < v < 1.$$

Alternatively, we may use the Lemmas stated on pages 10-11 of notes10.pdf. The joint density has the form $cp(u)q(v) = 24 \cdot 1/u^5 \cdot v(1-v)$ *on a support which is a product set* $A \times B = (1,\infty) \times (0,1)$. *We conclude that* U *and* V *are independent and that the density of* V *is proportional to* $q(v) = v(1-v)$ *on* $B = (0,1)$ *which leads to the answer given earlier.*

You are NOT required to show any work for the problems on this page. You will receive full credit just for filling in the blanks correctly. Some space is provided in case you should need it.

Problem 5. (4%) Suppose X, Y are iid with common density $f(x) = 2x$ for $0 < x < 1$ (and $f(x) = 0$ otherwise). The pdf of $Z = X + Y$ can be computed as

$$f_Z(z) = \begin{cases} \displaystyle\int_A^B 4x(z-x)\,dx & \text{for } 0 < z \leq 1, \\ \displaystyle\int_C^D 4x(z-x)\,dx & \text{for } 1 < z \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

What are the correct values for A, B, C, D that appear above? (Fill in the blanks.)

Give one point for each correct response.

A=__________ B=__________ C=__________ D=__________

Answers: $A = 0,\ B = z,\ C = z - 1,\ D = 1$

This is similar to the example on page 21 of notes10.pdf.

Problem 6. (6%) Suppose $X \sim N(2, 3)$ and $Y \mid X \sim N(X, 5)$. What is the marginal distribution of Y? (Fill in the blanks below. The first blank should contain the name of a distribution, and the second and third blanks should contain numbers.)

The marginal for Y has a

____________ distribution with mean = ______ and variance = ______.

The answers are: Normal, mean = 2, variance = 8.

We know that the joint distribution of (X, Y) *is bivariate normal from the discussion on page 12 of notes12.pdf. Therefore, the marginal distribution of* Y *is normal. The mean and variance may be computed quickly using the iterated expectations and double variance formulas on page 1 of notes11.pdf.*

Give two points for each correct response.