TEST #1
STA 5326 **Name:**____________________________________
October 4, 2016

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- **No work is required** for the problems marked **NWR**. For these problems, you will receive full credit just for stating the correct answer.

  This applies to part (a) of Problem 1 (but NOT to part (b)) and to Problems 7 and 8. You may show work on these problems if you wish, and you might receive partial credit for this work if your final answer is wrong.
- For all other problems, you must show and explain your work (including your calculations). **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- All the work on the exam should be your own. No "cooperation" is allowed.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **8** problems and **10** pages. There are a total of **100** points.

Problem 1.
In the game of Scrubble, players randomly draw six letters one by one from a large bag of letters. This bag is very large (essentially infinite) and contains all 26 letters with equal frequency. The players then earn points by spelling words with the letters they have drawn.

(a) (6%) **(NWR)** How many **different** possible draws of six letters are there? (Here two draws which have the same letters in a different order are considered the same.)

This problem is similar to exercises 1.19 and B2(b).

The answer is $\binom{26+6-1}{6} = \binom{31}{6}$.

This uses the counting rule for "unordered with replacement" in Table 1.2.1.

[**Problem 1 continued**]

(b) (12%) What is the probability that a player's draw of six letters will contain exactly two pairs? (For example, ABCDBA or ZCEEAC.)

Note: This part does **not** use the answer to part (a).

This problem is similar to 1.46 because it can be restated as: "Six balls are distributed randomly into 26 cells. Find the probability there are two cells containing exactly two balls." The problem is also somewhat similar to the poker problem (probability of a full house) in lecture and to the first of the solutions given for exercise 1.20.

The answer is:

$$\frac{\binom{26}{2}\binom{24}{2}\binom{6}{2}\binom{4}{2}2!}{26^6} = \left\{\binom{6}{2}\binom{4}{2}\cdot\frac{1}{2}\right\}\left[\frac{26}{26}\cdot\frac{1}{26}\cdot\frac{25}{26}\cdot\frac{1}{26}\cdot\frac{24}{26}\cdot\frac{23}{26}\right]$$

Solution: Since the letter bag is "very large" and the letters occur with equal frequency, the draws from the bag are independent and each letter has probability $\frac{1}{26}$ on each draw. So this situation is the same as the "monkey" examples in lecture except here the monkey is typing 6 letters at random.

The left-hand answer is obtained by a counting argument. As in the monkey examples, we know there are 26^6 equally likely 6 letter sequences, which explains the denominator. If there are exactly two pairs in the sequence, then there are two letters which occur twice in the sequence, and another two letters which occur only once. We can construct such a sequence in the following steps. (1) Choose the two letters which will occur twice. (2) Choose the two letters which will occur once. (3) Choose two positions out of the six for the first letter (i.e., first in alphabetical order) chosen in step 1. (4) Choose two positions for the second letter chosen in step 1. (5) Place the two letters chosen in step 2 in the two remaining positions. The five factors in the numerator give the number of ways each of these steps can be performed.

The right-hand answer is obtained by a combination of probability arguments and counting. One way to obtain exactly two pairs is to get a sequence of the form AABBCD, that is, the second letter drawn must be the same as the first, the third letter must be different from the first two, the fourth letter must be the same as the third, the fifth letter must be different from all the previous letters, and the sixth letter must be different from all the previous letters. This is an intersection of five events $A_1, A_2, \ldots, A_5$ whose probability is given by the product $P(A_1)P(A_2|A_1)\ldots P(A_5|A_1 \cap A_2 \cap A_3 \cap A_4)$ whose factors are the five fractions inside the square brackets (after excluding $\frac{26}{26}$, which is included just to make the answer look nicer, and represents the fact that the first letter can be anything).
We have obtained the probability of getting two pairs in the order AABBCD, but there are other orders which could occur, all of which have the same probability. The number of possible such orderings is the quantity in curly braces: there are $\binom{6}{2}$ ways to pick the two positions occupied by the singleton letters (represented by C and D), and $\binom{4}{2}\cdot\frac{1}{2}$ ways to divide the remaining four positions into the two pairs (represented by A and B). The factor of $\frac{1}{2}$ arises because we can interchange the letters A and B, that is, dividing four positions into AABB is equivalent to BBAA, etc.; they both give the same division of the four positions into two groups of two positions.

Problem 2. (12%) Suppose the random variable X has pdf

$$f(x) = \begin{cases} \frac{4-x}{12} & \text{for } -1 < x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find a monotone function g such that $Y = g(X)$ has a Uniform(0,1) distribution.

This is similar to exercise 2.9.

Note: $g(x)$ must be defined and correct for $-1 < x < 3$, but outside of that range it can assume arbitrary values since X only takes values in $(-1, 3)$.

Problem 3. (12%) A hat contains two biased coins. Coin #1 has $P(\text{heads}) = 1/3$ and Coin #2 has $P(\text{heads}) = 2/3$. A coin is selected at random from the hat and tossed two times. (The **same** coin is tossed two times.)

If at least one of the tosses is heads, what is the probability that both tosses are heads?

This combines features of exercise 1.25 with the examples in notes1.pdf involving tossing a randomly selected coin.

Let $C = \{\textit{at least one of the tosses is heads}\}$ *and* $D = \{\textit{both tosses are heads}\}$. *Let* $B_1 = \{\textit{select coin \#1}\}$ *and* $B_2 = \{\textit{select coin \#2}\}$. *Then*

$$\begin{aligned}
P(D|C) &= \frac{P(D \cap C)}{P(C)} = \frac{P(D)}{P(C)} \\
&= \frac{P(D|B_1)P(B_1) + P(D|B_2)P(B_2)}{P(C|B_1)P(B_1) + P(C|B_2)P(B_2)} \\
&= \frac{P(D|B_1) + P(D|B_2)}{P(C|B_1) + P(C|B_2)} \quad \textit{since } P(B_i) = 1/2 \textit{ for } i = 1, 2 \\
&= \frac{(1/3)^2 + (2/3)^2}{(1 - (2/3)^2) + (1 - (1/3)^2)} \\
&= \frac{5/9}{13/9} = \frac{5}{13}\,.
\end{aligned}$$

Problem 4. An urn contains 23 balls. Seven of these balls are red.

(a) (12%) Suppose you randomly sample 5 balls with**OUT** replacement. Let X be the number of red balls in your sample. Find an expression for $P(X = k)$. (This expression should be valid for $k = 0, 1, \ldots, 5$.)

This is similar to exercise 1.51.

[**Problem 4 continued**]
An urn contains 23 balls. Seven of these balls are red.

(b) (12%) Suppose now that you randomly draw balls from this urn (one by one and with**OUT** replacement) until you finally draw a red ball. Let Y be the total number of balls that you draw. Find $P(Y > 4)$.

This exercise is similar to 1.26.

Let $A_i = \{\text{ball } i \text{ is not red}\}$.

$$\begin{aligned} P(Y > 4) &= P(\text{no red balls in the first four draws}) \\ &= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)P(A_4|A_1 \cap A_2 \cap A_3) \\ &= \frac{16}{23} \cdot \frac{15}{22} \cdot \frac{14}{21} \cdot \frac{13}{20}. \end{aligned}$$

Problem 5. (12%) Suppose X has pdf $f_X(x) = x/8$ for $0 < x < 4$ (and $f_X(x) = 0$ otherwise). Find $P(1/3 < Y < 2/3)$ where $Y = g(X)$ and g is defined as follows:

$$g(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 1/2 & \text{for } 1 \le x \le 3 \\ x - 3 & \text{for } 3 < x < 4. \end{cases}$$

This is similar to the lecture example in notes3.pdf, pages 3 and 4.

Problem 6. (12%) Suppose X has pdf $f_X(x) = \frac{1}{4}(x+1)^3$ for $-1 < x < 1$. Find the pdf of $Y = X^2 + 1$.

This is similar to exercise 2.6(b). A correct answer should clearly state the interval in which the density $f_Y(y)$ is nonzero, which is $1 < y < 2$.

Problem 7. (5%) **(NWR)** Consider a modified "Gambler's Ruin" problem where the game consists of a sequence of rolls of a **fair die**. When the die comes up 1 or 2, the player **wins** \$1. When the die comes up 3, the player gets \$0 (nothing). When the die comes up 4, 5 or 6, the player **loses** \$1. Let $\psi(z)$ denote the probability of reaching a given goal of g dollars starting with an initial fortune of z dollars where $0 < z < g$. Use the Law of Total Probability to find an equation that $\psi(z)$ must satisfy.

Answer: $\psi(z) = \frac{1}{3}\psi(z+1) + \frac{1}{6}\psi(z) + \frac{1}{2}\psi(z-1)$ *or anything algebraically equivalent to this such as* $\psi(z) = \frac{2}{5}\psi(z+1) + \frac{3}{5}\psi(z-1)$.

Problem 8. (5%) **(NWR)** A dart is thrown uniformly at random on a circular target with radius R. Let Z be the distance of this dart from the center of the target. Give an expression for $P_Z\big((a,b)\big)$. (This expression should be valid for $0 < a < b < R$.)

See notes2.pdf, page 14.