TEST #3
STA 5326 **Name:**____________________________________
December 8, 2016

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items including (and especially) CELL PHONES must be left at the front of the room.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- For all problems you must show and explain your work (including your calculations). **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density (or joint density) is zero outside of some interval (or region), this interval (or region) should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- All the work on the exam should be your own. No "cooperation" is allowed.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- The exam has **5** problems and **12** pages. There are a total of **100** points.

Problem 1. (15%) Let (X, Y) be a bivariate random vector with joint pdf $f(x, y)$ which is positive on the entire plane R^2. Let

$$U = aX + bY + c \quad \text{and} \quad V = aX - bY$$

where a, b, c are fixed constants with $a > 0$ and $b > 0$. Find the joint pdf of (U, V).

This is (sort of) a combination of exercises 4.22 and 4.27.

This bivariate transformation is 1-1 from R^2 to R^2. The inverse transformation is

$$X = \frac{1}{2a}(U + V - c)\,, \quad Y = \frac{1}{2b}(U - V - c)\,.$$

The Jacobian of the inverse transformation is $\dfrac{-1}{2ab}$ with absolute value $\dfrac{1}{2ab}$. Therefore the answer is

$$f_{U,V}(u, v) = f\left(\frac{1}{2a}(u + v - c), \frac{1}{2b}(u - v - c)\right) \cdot \frac{1}{2ab}\,.$$

Problem 2. A pdf is defined by

$$f(x,y) = \begin{cases} 8xy & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

[Note: This pdf is positive on a triangle with corner points $(0,0)$, $(1,0)$, and $(1,1)$.]

This problem is similar to exercise 4.4 and to the joint density on a triangle example used in notes9.pdf.

(a) (6%) Find $f_Y(y)$, the marginal density of Y.

The answer is $f_Y(y) = 4y(1-y^2)$ *for* $0 < y < 1$.

Solution:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx = \int_y^1 8xy\,dx \\ &= 8y \int_y^1 x\,dx = 4y \cdot x^2\big|_y^1 = 4y(1-y^2) \quad \textit{for } 0 < y < 1. \end{aligned}$$

$f_Y(y) = 0$ *otherwise since then the horizontal line* $Y = y$ *does not intersect the triangle* $0 < y < x < 1$.

[**Problem 2 continued**]

(b) (5%) Are X and Y independent? Answer "Yes" or "No" and justify your answer.

The answer is "No, they are not independent".

This can be justified in a few ways. The simplest is to note that the support of the joint density $f(x, y)$ *is* $0 < y < x < 1$*, which is NOT a product set. Therefore,* X *and* Y *canNOT be independent.*

The second approach is to show that the statement

$$f(x, y) = f_X(x) f_Y(y) \text{ for all } x \text{ and } y$$

is false. We already know $f_Y(y) = 4y(1 - y^2)$ *for* $0 < y < 1$*. A similar calculation shows* $f_X(x) = 4x^3$ *for* $0 < x < 1$*. So clearly the statement above is false since* $f_X(s) f_Y(y) = 16x^3 y(1-y^2)$ *for* $0 < x < 1, 0 < y < 1$*.*

[**Problem 2 continued**]

(c) (6%) Let $F(x, y)$ denote the joint **c**df of (X, Y). Evaluate $F(1/2, 3/4)$.

The answer is $F(1/2, 3/4) = 1/16$.

It is easily seen that $F(1/2, 3/4) = F_X(1/2)$ *and that* $F_X(x) = x^4$ *for* $0 < x < 1$.

Further details:

$$\begin{aligned}
F(1/2, 3/4) &= P(X \le 1/2, Y \le 3/4) = P(X \le 1/2) \quad (\mathit{Drawthepicture!}) \\
&= \int_0^{1/2} \int_0^x 8xy \, dy \, dx = 8 \int_0^{1/2} x \int_0^x y \, dy \, dx \\
&= 4 \int_0^{1/2} x^3 \, dx = \frac{1}{16} \, .
\end{aligned}$$

[**Problem 2 continued**]

(d) (6%) Find $P(X > \sqrt{Y})$.

This resembles exercise 4.5. The answer is 2/3.
Solution:

$$P(X > \sqrt{Y}) = P(Y < X^2) = \int_0^1 \int_0^{x^2} 8xy\,dy\,dx$$
$$= 8\int_0^1 x \int_0^{x^2} y\,dy\,dx = 4\int_0^1 x^5\,dx = 2/3\,.$$

[**Problem 2 continued**]

(e) (6%) Suppose $0 < y < 1$. Find $f_{X|Y}(x|y)$ for all x.

This is similar to exercise 4.4(e), one of the additional parts of 4.4.

The answer is

$$f_{X|Y}(x|y) = \frac{2x}{1-y^2} \quad \textit{for } y < x < 1 \quad \textit{(and zero otherwise).}$$

[**Problem 2 continued**]

(f) (6%) Suppose $0 < y < 1$. Find $E(X|Y = y)$.

This is similar to exercise 4.4(f), one of the additional parts of 4.4.

The answer is:

$$E(X|Y = y) = \frac{2}{3} \cdot \frac{1 - y^3}{1 - y^2} = \frac{2}{3} \cdot \frac{y^2 + y + 1}{y + 1} \quad \textit{for } 0 < y < 1 \textit{ (and undefined otherwise).}$$

Problem 3. (15%) Let X and Y be independent random variables with densities

$$f_X(x) = \frac{e^{-x^2}}{\sqrt{\pi}}\,, \quad -\infty < x < \infty \quad \text{and} \quad f_Y(y) = e^{-y}\,, \;\; 0 \le y < \infty$$

and define

$$U = X^2 + Y \quad \text{and} \quad W = \frac{X}{\sqrt{X^2+Y}}\,.$$

Find the joint density of U and W.

This is similar to exercise 4.20. The situation is a little simpler than 4.20 since the transformation is one-to-one so that the factor of 2 in the answer of 4.20 is not needed here.

The answer is:

$$f_{U,V}(u,v) = \pi^{-1/2}u^{1/2}e^{-u} \quad \textit{for } 0 < u < \infty, -1 < w < 1.$$

A few details:

The support of (U, W) is $(0,\infty) \times (-1, 1)$.

The inverse transformation is:

$$X = W\sqrt{U}$$
$$Y = U(1 - W^2)$$

The Jacobian is $\sqrt{U}$.

Note: The problem does NOT request this, but from the answer we easily see that U and V are independent with $U \sim Gamma(3/2, 1)$ and $V \sim Uniform(-1, 1)$.

Problem 4. The random pair (X, Y) has the distribution (joint mass function)

		X			
		1	2	3	4
	1	0.25	0.12	0.16	0.07
Y	2	0.05	0.18	0.04	0.13

This is similar to exercise 4.10.

(a) (7%) Show that X and Y are dependent.

The marginals are:

x	*1*	*2*	*3*	*4*
$f_X(x)$	*.30*	*.30*	*.20*	*.20*

y	*1*	*2*
$f_Y(y)$	*.60*	*.40*

The random variables X and Y are independent if and only if the joint pmf satisfies $f(x, y) = f_X(x) f_Y(y)$ for all x and y. This actually fails for every entry of the table for $f(x, y)$. For example,

$$f(1,1) = 0.25 \neq 0.18 = (0.30)(0.60) = f_X(1) f_Y(1)\,.$$

(b) (7%) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

[**Problem 4 continued**]

(c) (6%) Calculate $E(X|Y = 2)$. (Use the original probability table given in the problem statement.)

See pages 8, 11 and 14-15 of notes9.pdf.

Problem 5. (15%) Let X and Y be independent random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2. Find an expression for $\mathrm{Cov}(3X^2Y, Y+5)$, the **covariance** between $3X^2Y$ and $Y+5$.

Similar to exercise 4.42. See also pages 1 and 2 of notes12.pdf.

The answer is:

$$\begin{aligned} Cov(3X^2Y, Y+5) &= 3\,Cov(X^2Y, Y) \\ &= 3(E(X^2Y^2) - (EX^2Y)(EY)) \\ &= 3((EX^2)(EY^2) - (EX^2EY)(EY)) \\ &= 3((\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - (\sigma_X^2 + \mu_X^2)\mu_Y^2) \\ &= 3(\sigma_X^2 + \mu_X^2)\sigma_Y^2 \,. \end{aligned}$$