TEST #1
STA 5326 **Name:**____________________________________
October 3, 2017

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems. **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- **If your answer is valid only for a certain range of values, this should be stated as part of your answer. For example, if a density is zero outside of some interval, this interval should be stated explicitly.**
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **7** problems and **9** pages. There are a total of **100** points.

Problem 1. (13%) Joe has **8** friends who are coming to visit him. They are arriving during a period of **5** days, with each friend arriving on a randomly chosen day. What is the probability that there are **2** days on which **3** friends arrive?

This is similar to exercise 1.46.

The event $A = \{$*there are 2 days on which 3 friends arrive*$\}$ *can happen in two ways:* $\{332\}$ *and* $\{3311\}$*. Here* $\{3311\} = \{$*2 days on which 3 friends arrive, and 2 days on which 1 friend arrives*$\}$ *and* $\{332\} = \{$*2 days on which 3 friends arrive, and 1 day on which 2 friends arrive*$\}$*.*
There are 5^8 *equally likely ways to assign the 8 friends to the 5 days.*
$P(332) = \frac{\binom{5}{2}\binom{3}{1}\cdot\binom{8}{3}\binom{5}{3}}{5^8} = \frac{16800}{390625} = 0.043008$

Explanation: $\binom{5}{2}\binom{3}{1}$ *is the number of ways to choose the days (2 days on which 3 friends are to arrive, and 1 day for the remaining 2 friends).* $\binom{8}{3}\binom{5}{3}\binom{2}{2}$ *is the number of ways to choose the people to arrive on those days.*

$P(3311) = \frac{\binom{5}{2}\binom{3}{2}\cdot\binom{8}{3}\binom{5}{3}\binom{2}{1}}{5^8} = \frac{33600}{390625} = 0.086016$

Explanation: $\binom{5}{2}\binom{3}{2}$ *is the number of ways to choose the days.* $\binom{8}{3}\binom{5}{3}\binom{2}{1}\binom{1}{1}$ *is the number of ways to choose the people to arrive on those days.*

So $P(A) = 0.043008 + 0.086016 = 0.129024$. *(Getting a decimal answer is not required.)*

Problem 2. (13%) Every year at the county fair a competition is held to find the heaviest pumpkins, and prizes are given to the first, second, and third place pumpkins. Suppose n pumpkins are entered in the competition and the pumpkins are weighed in a random order. If the i^{th} pumpkin weighed is heavier than the previous $i-1$ (it is best so far), what is the probability that the i^{th} pumpkin ends up in **second** place?

This is similar to 1.32, but a little harder.

Let $A = \{i\text{-th pumpkin gets 2nd place}\}$ *and* $B = \{i\text{-th pumpkin is better than previous } i-1\}$. *We want to calculate* $P(A|B)$.

This may be done in a couple of ways.

First approach: Note that $P(A|B) = \frac{P(A)P(B|A)}{P(B)}$. $P(B) = 1/i$ *since, by random ordering, all of the first* i *pumpkins are equally likely to be the best among the first* i. *Since all of the pumpkins are equally likely to be placed in position* i, *we know* $P(A) = 1/n$. *Given* A *has occurred,* B *occurs only if the first place pumpkin is after* i *in the order, that is, it must be in one of the* $n-i$ *positions* $i+1, i+2, \ldots, n$. *Thus* $P(B|A) = (n-i)/(n-1)$. *Here the denominator is* $n-1$ *since, given* A, *there are only* $n-1$ *remaining possible positions (all equally likely) for the first place pumpkin. Thus the answer is* $\dfrac{P(A)P(B|A)}{P(B)} = \dfrac{(1/n)\cdot(n-i)/(n-1)}{(1/i)} = \dfrac{i(n-i)}{n(n-1)}$.

Second approach: There are $n!$ *possible ways to order the pumpkins, all equally likely.* $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\#(A\cap B)}{\#(B)}$ *since all outcomes are equally likely. We find* $\#(B)$ *by counting the number of orderings in which pumpkin* i *is best so far. We may construct such an ordering in three steps. First: select* i *pumpkins to be the first* i. *This can be done in* $\binom{n}{i}$ *ways. Second: place the heaviest of these* i *pumpkins in position* i *and order the remaining* $i-1$ *arbitrarily in the remaining* $i-1$ *positions* $1, 2, \ldots, i-1$. *This can be done in* $(i-1)!$ *ways. Third: place the remaining* $n-i$ *pumpkins arbitrarily in positions* $i+1, i+2, \ldots, n$. *This can be done in* $(n-i)!$ *ways. Thus,* $\#(B) = \binom{n}{i}(i-1)!(n-i)! = \frac{n!}{i}$. *The event* $A\cap B$ *occurs only if the second place pumpkin is in position* i *and the first place pumpkin is in one of the* $n-i$ *later positions* $i+1, i+2, \ldots, n$. *Constructing such an arrangement can be done is two steps. Initially, we place the second place pumpkin in position* i. *(We have no choice about this, so I will not count it as a step.) First: we place the first place pumpkin in one of the positions* $i+1, \ldots, n$. *This can be done in* $n-i$ *ways. Second: we place the remaining* $n-2$ *pumpkins arbitrarily in the remaining* $n-2$ *positions. This can be done in* $(n-2)!$ *ways. Thus* $\#(A\cap B) = (n-i)\cdot(n-2)!$. *So our answer is* $\dfrac{\#(A\cap B)}{\#(B)} = \dfrac{(n-i)\cdot(n-2)!}{n!/i} = \dfrac{i(n-i)}{n(n-1)}$.

Problem 3. (13%) A coin has probability π of heads on any toss. Suppose **three** people each toss this coin n times. What is the probability all three of them get the same number of heads?

(Your answer does **NOT** have to be simplified.)

This is similar to exercise 1.23.

Let X, Y, Z be the number of heads tossed by the three people. X, Y, Z are independent Binomial(n, π) rv's. Therefore

$$P(X = Y = Z) = \sum_{i=0}^{n} P(X = Y = Z = i) = \sum_{i=0}^{n} [P(X = i)]^3 = \sum_{i=0}^{n} \left[\binom{n}{i} \pi^i (1-\pi)^{n-i} \right]^3 .$$

Problem 4. (12%) Suppose there are 10 multiple choice questions, each with 5 responses. If you guess on all 10 questions, what is the probability you get at least two correct answers?

This is similar to exercises 1.36 and 1.44.

Problem 5. (12%) Let $A_1, A_2, \ldots, A_n$ be arbitrary sets. One of DeMorgan's Laws states that:

$$\left(\bigcup_{i=1}^{n} A_i\right)^{\boldsymbol{c}} = \underline{\hspace{3cm}}$$

Fill in the blank above with the expression which correctly completes this equation. Then give a clear proof of this statement in the space below.

Proof:

This is part of Exercise 1.10.

Problem 6. (13%) Suppose a monkey types **4** digits at random. (The key strokes are independent with each of the **10** digits 0123456789 having equal probability **1/10**.)

What is the probability the monkey types **99**? (This means that two consecutive digits 99 appear somewhere in the monkey's four typed digits.)

This is similar to the monkey examples on pages 13 and 14 of notes1.pdf

Problem 7. Let X have the density $f_X(x) = \dfrac{x^2}{9}$ for $0 < x < 3$.

(a) (12%) Find the density of $Y = 5 + X^3$.

This is a standard problem of finding the density of $Y = g(X)$ *when* g *is smooth and monotonic. For examples, see exercises 2.1 and 2.2. The answer is* $f_Y(y) = 1/27$ *for* $5 < y < 32$ *(and zero otherwise). Thus* Y *has a uniform distribution on the interval* $(5, 32)$*. Stating the interval* $(5, 32)$ *where the density is nonzero is a very important part of the answer.*

[**Problem 7 continued**]

Recall: X has density $f_X(x) = \dfrac{x^2}{9}$ for $0 < x < 3$.

(b) (12%) Find the density of $Y = g(X)$ where

$$g(x) = \begin{cases} 4x & \text{for } x < 2 \\ 18 - 5x & \text{for } x \geq 2. \end{cases}$$

This is a simplified version of the example on pages 27-29 of notes3.pdf. It can be worked by the approach on page 26 of notes3.pdf or by the approach on page 24 of notes3.pdf (or from first principles).
The approach on page 26 of notes3.pdf leads to the answer in this form:

$$f_Y(y) = \frac{(y/4)^2}{9} \cdot \frac{1}{4} \cdot I_{(0,8)}(y) + \frac{((18-y)/5)^2}{9} \cdot \frac{1}{5} \cdot I_{(3,8)}(y)$$

If this answer is left in this form, that is OK. Give them full credit.
It is also possible to simplify this answer to be:

$$f_Y(y) = \begin{cases} \dfrac{21}{8000}y^2 - \dfrac{4}{125}y + \dfrac{36}{125} & 3 \leq y < 8 \\ \dfrac{1}{576}y^2 & 0 < y < 3 \end{cases}$$

The precise definitions at the endpoints of the cases is not important.