TEST #2
STA 5326 **Name:**__________________________________
November 7, 2017

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems (**except those on the last page**). **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and **8** pages. There are a total of **100** points.

Problem 1. (15%) A town has 40,000 residents. These people fall into 4 categories, each containing 10,000 people. For a person in each of these categories, the probability of suffering a fatal heart attack in the next year is given in the following table:

Category	#1	#2	#3	#4
Probability	1×10^{-4}	2×10^{-4}	3×10^{-4}	4×10^{-4}

Assuming the people are independent, what is the probability that exactly 5 people in this town suffer a fatal heart attack in the next year? (Use a suitable approximation.)

This is similar to exercise C2. The easiest solution is to use a general Poisson approximation as on page 30 of notes6.pdf and in exercise C2. Another approach is to use the Poisson approximation to the binomial for the number of heart attacks in each category, and then use the closure property of the Poisson distribution to add the categories and get an approximate Poisson distribution for the total number of fatal heart attacks.

Problem 2. Al and Bob each roll a fair die 72,000 times and keep track of the number of 1's they get.

(a) (14%) Find a value k such that the probability of Al getting more than k 1's is approximately 0.025.

This is similar to Exercise 3.8. A normal approximation gives $k \approx 12,196$.

[**Problem 2 continued**]
Al and Bob each roll a fair die 72,000 times and keep track of the number of 1's they get.

(b) (15%) What is the probability Al and Bob get exactly the same number of 1's? (Compute a numerical answer using an appropriate approximation.)

Show the argument and do NOT just quote a result from lecture!

This is similar to exercise C5.

Problem 3. (15%) Consider a sequence of independent tosses of an old-fashioned toothpaste cap. (Many brands no longer use them.) A toothpaste cap has an irregular shape and will land on its Top, Bottom, or Side with probabilities p, q, and r, respectively, where $p + q + r = 1$. Define a random variable X as the length of the run (of either Top, Bottom, or Side) started by the first toss. (For example, $X = 3$ if either $TTTB$, $TTTS$, $BBBT$, $BBBS$, $SSST$, or $SSSB$ is observed.)

Find EX.

Similar to Exercise 2.13. The answer is:

$$\frac{p}{1-p} + \frac{q}{1-q} + \frac{r}{1-r}$$

which (unfortunately) can be written in many other ways.

Problem 4. (15%) Suppose the random variable X has density (pdf) given by

$$f(x) = \begin{cases} \frac{2}{3}e^{-x} & x \geq 0 \\ \frac{2}{3}e^{2x} & x < 0\,. \end{cases}$$

Compute the moment generating function (mgf) of X.
(Make sure to specify the range where the mgf is well-defined and finite.)

This is similar to finding the mgf of the Double Exponential distribution (also called the Laplace distribution) done on page 4 of notes5.pdf.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{0} e^{tx}\cdot\frac{2}{3}\,e^{2x}\,dx + \int_{0}^{\infty} e^{tx}\cdot\frac{2}{3}\,e^{-x}\,dx$$

$$= \frac{2}{3}\int_{-\infty}^{0} e^{(t+2)x}\,dx + \frac{2}{3}\int_{0}^{\infty} e^{(t-1)x}\,dx$$

$$= \begin{cases} \frac{2}{3}\,\frac{e^{(t+2)x}}{t+2}\Big|_{x=-\infty}^{x=0} & \textit{if } t+2>0 \\ \infty \textit{ or undefined} & \textit{otherwise} \end{cases} + \begin{cases} \frac{2}{3}\,\frac{e^{(t-1)x}}{t-1}\Big|_{x=0}^{x=\infty}\,dx & \textit{if } t-1<0 \\ \infty \textit{ or undefined} & \textit{otherwise} \end{cases}$$

$$= \begin{cases} \frac{2}{3}\cdot\frac{1}{t+2} & \textit{if } -2<t \\ \infty & \textit{otherwise} \end{cases} + \begin{cases} \frac{2}{3}\cdot\frac{(-1)}{t-1} & \textit{if } t<1 \\ \infty & \textit{otherwise} \end{cases}$$

$$= \begin{cases} \frac{2}{3}\cdot\frac{1}{t+2} + \frac{2}{3}\cdot\frac{1}{1-t} & \textit{if } -2<t<1 \\ \infty & \textit{otherwise} \end{cases}$$

$$= \begin{cases} \frac{2}{(t+2)(1-t)} & \textit{if } -2<t<1 \\ \infty & \textit{otherwise} \end{cases}$$

Problem 5. (14%) If $X \sim \text{Gamma}(\alpha, \beta)$ and α is large, give an argument to show that X is approximately Normal.

See page 23 of notes7.pdf.

For the problems on this page, you will receive full credit just for giving the correct answer. No work is required.

Problem 6. (4%) Suppose r is a positive integer and $t > 0$. The integral below can be evaluated and shown to be equal to a certain summation. Write this summation on the blank provided. (Hint: The fact that the integral equals the summation can be interpreted as a relation between T_r = "the time of the r-th arrival" and S_t = "the number of arrivals in the time interval $(0, t)$" for a Poisson process of constant rate $\lambda = 1$.)

$$\int_t^\infty \frac{1}{\Gamma(r)} x^{r-1} e^{-x}\, dx = \underline{\hspace{4cm}}$$

This is Exercise 3.19. Also given on page 16 of notes7.pdf.

Problem 7. (4%) An urn contains R red balls and G green balls. A sample of k balls are drawn from this urn with**OUT** replacement. The number of ordered samples of k balls is $k!\binom{R+G}{k}$. For $i \neq j$, what is the **number** of ordered samples of k balls in which the i^{th} and j^{th} balls are red?

See page 24 of notes6.pdf. Unfortunately, this expression can be written in many equivalent ways.

Problem 8. (4%) Let $f(x)$ be the Cauchy density: $f(x) = \dfrac{1}{\pi}\dfrac{1}{1+x^2}, \quad -\infty < x < \infty$

This density is symmetric about zero. The value of $\int_{-A}^{A} f(x)\,dx$ is given for some values of A in the table below:

A	0.1	1	10
$\int_{-A}^{A} f(x)\,dx$	0.063451	0.5	0.93655

Let $X_1, X_2, X_3, \ldots$ be iid random variables having the Cauchy density.

Suppose $n = 10^{100}$ and $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$.

Then we know that $P(-1 < \overline{X} < 1)$ ________________.

Circle the response which best fills the blank above.

a) is close to zero **b)** equals 0.063451 **c)** equals 0.093655 **d)** equals 0.5

e) equals 0.63451 **f)** equals 0.68269 **g)** equals 0.93655 **h)** is close to one

We know $P(-1 < \overline{X} < 2) = 0.5$ by the remark on page 6 of notes4.pdf that $\overline{X}$ has the same distribution as X_1.