TEST #3
STA 5326 **Name:**____________________________________
December 7, 2017

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems (**except those on the last page**). **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **9** problems and **10** pages. There are a total of **100** points.

Problem 1. Suppose that X has density $12x^2(1-x)$ for $0 < x < 1$, and $Y|X$ has a Geometric distribution with parameter $p = X$.

(a) (10%) Find EY.

[**Problem 1 continued**]

(b) (12%) Find $\text{Var}(Y)$.

[**Problem 1 continued**]

(c) (12%) Find the marginal mass function (pmf) of Y.

Problem 2. (11%) Let X and Y be independent Binomial(n, p) random variables. Find the conditional distribution of $Y \mid X + Y$. That is, if u and w are integers with $0 \leq w \leq u$, find $P(Y = w \mid X + Y = u)$.

(You may use without proof the fact that $X + Y \sim$ Binomial$(2n, p)$.)

Problem 3. (12%) Let X and Y be independent random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2. Find an expression for $\mathrm{Cov}(X+Y, XY)$, the covariance between $X+Y$ and XY, in terms of these means and variances.

Problem 4. (10%) Find $P(Y^2 < X < Y)$ if X and Y are jointly distributed with pdf

$$f(x,y) = 2x\,, \quad 0 \le x \le 1, \quad 0 \le y \le 1\,.$$

Problem 5. (12%) Let X and Y be independent with densities

$$f_X(x) = e^{-x}\,,\ x \geq 0 \qquad \text{and} \qquad f_Y(y) = \frac{e^{-1/y}}{y^2}\,,\ y \geq 0.$$

Find the density of XY.

Problem 6. (5%) Here is a joint density function:

$$f_{X,Y}(x,y) = \frac{1}{2\pi(1-e^{-8})} e^{-x^2/2} \sin^2(y) e^{-y^2/8}, \quad -\infty < x < \infty, \ -\infty < y < \infty .$$

Find the marginal density for X. (Simplify as much as possible.)

The remaining problems require no work. You will receive full credit for stating the correct answer.

Problem 7. Suppose $X_0, X_1, X_2, \ldots$ is a Markov chain with state space $S = \{1, 2, \ldots, m\}$, initial probability distribution given by the row vector $\boldsymbol{a}$ with entries $a_i = P(X_0 = i)$, and transition probability matrix $\boldsymbol{P}$.

(a) (5%) Give a general expression for $P(X_n = k)$. (Use matrix notation and state your answer as a specific entry in a specific vector or matrix for which you give a formula.)

$P(X_n = k) =$ ______________________________

(b) (5%) Suppose there are 4 states ($m = 4$) and that the chain is irreducible and aperiodic with stationary distribution $\boldsymbol{\pi} = (0.4, 0.3, 0.2, 0.1)$. Suppose we compute the matrix product $\boldsymbol{P}^h$ where $\boldsymbol{P}$ is the transition probability matrix and h is a huge number (say, $h = 10^{100}$). Then $\boldsymbol{P}^h$ will be approximately equal to what matrix? (Write this matrix explicitly in the space provided.)

$\boldsymbol{P}^h \approx$ ______________________________

Problem 8. (3%) Simplify the expression given below. Write your answer in the space provided:

$$\frac{d}{dx}\int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} g(u,y)\,dy\right)du \;=\; \underline{\hspace{6cm}}$$

(Assume the function g is such that the integrals are finite and the derivative exists and all necessary regularity conditions are true.)

Problem 9. (3%) For a Markov process,

$$P(X_{n+1} = i_{n+1} \,|\, X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0) = \ldots$$

Circle the response which correctly completes this equation.

a) $P(X_{n+1} = i_1 \,|\, X_n = i_0)$ **b**) $P(X_{n+1} = i_{n+1} \,|\, X_n = i_n)$ **c**) $P(X_1 = i_1 \,|\, X_0 = i_0)$

d) $P(X_{n+1} = i_{n+1} \,|\, X_0 = i_0)$ **e**) $P(X_{n+1} = i_{n+1} \,|\, X_0 = i_n)$ **f**) $P(X_0 = i_n \,|\, X_1 = i_{n+1})$