TEST #3
STA 5326 **Name:**__________________________________
December 7, 2017

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems (**except those on the last page**). **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **9** problems and **10** pages. There are a total of **100** points.

General Remark: I do NOT require students to simplify their arithmetic. They can leave in fractions, powers, factorials, etc. If you can see that their answer is correct, give them full credit. But students **must** *do all the necessary algebra and calculus to get full credit. They must compute all derivatives; they should lose points if they leave 'd/dx's or 'primes' in their answer. Also, student must simplify summations if there is a simple closed form.*

I usually do not deduct points for arithmetic errors unless they make the answer ridiculous. (For example, probabilities not between 0 and 1, negative expected values for positive-valued random variables, negative variances, etc.).

I usually do not deduct points for copying errors (unless they make the answer ridiculous).

Problem 1. Suppose that X has density $12x^2(1-x)$ for $0 < x < 1$, and $Y|X$ has a Geometric distribution with parameter $p = X$.

Similar to exercise 4.31.

(a) (10%) Find EY.

$$EY = EE(Y|X) = E\left(X^{-1}\right) = \int_0^1 x^{-1} \cdot 12x^2(1-x)\,dx = 12\int_0^1 x(1-x)\,dx = 2$$

[**Problem 1 continued**]

(b) (12%) Find Var(Y).

$$
\begin{aligned}
Var(Y) &= E\,Var(Y|X) + Var(E(Y|X)) \\
&= E\left(\frac{1-X}{X^2}\right) + Var(X^{-1}) \\
&= \int_0^1 (1-x)x^{-2} \cdot 12x^2(1-x)\,dx + (EX^{-2} - (EX^{-1})^2) \\
&= 12\int_0^1 (1-x)^2\,dx + 12\int_0^1 (1-x)\,dx - \left(12\int_0^1 x(1-x)\right)^2 \\
&= 4 + 6 - 4 = 6
\end{aligned}
$$

[**Problem 1 continued**]

(c) (12%) Find the marginal mass function (pmf) of Y.

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_X(x) f_{Y|X}(y|x)\,dx \\
&= \int_0^1 12x^2(1-x)\cdot x(1-x)^{y-1}\,dx \\
&= 12\int_0^1 x^3(1-x)^y\,dx \\
&= 12B(4,y+1) = 12\cdot\frac{\Gamma(4)\Gamma(y+1)}{\Gamma(y+5)} = 12\cdot\frac{3!y!}{(y+4)!} \\
&= \frac{72}{(y+1)(y+2)(y+3)(y+4)} \quad \textit{for } y = 1,2,3,\ldots.
\end{aligned}$$

If students leave the answer in terms of gamma functions, that is OK and should get full credit.

Problem 2. (11%) Let X and Y be independent Binomial(n, p) random variables. Find the conditional distribution of $Y \mid X + Y$. That is, if u and w are integers with $0 \le w \le u$, find $P(Y = w \mid X + Y = u)$.

(You may use without proof the fact that $X + Y \sim$ Binomial$(2n, p)$.)

This problem is similar to Exercise 4.15.

The answer is $\frac{\binom{n}{w}\binom{n}{u-w}}{\binom{2n}{u}}$.

This is actually a hypergeometric pmf (as a function of w keeping u fixed), but students are **not** *required to mention this.*

Problem 3. (12%) Let X and Y be independent random variables with means μ_X, μ_Y and variances σ_X^2, σ_Y^2. Find an expression for $\mathrm{Cov}(X+Y, XY)$, the covariance between $X+Y$ and XY, in terms of these means and variances.

Similar to Exercise 4.42 with a little of 4.43 mixed in.

Problem 4. (10%) Find $P(Y^2 < X < Y)$ if X and Y are jointly distributed with pdf

$$f(x,y) = 2x\,, \quad 0 \le x \le 1, \quad 0 \le y \le 1\,.$$

Very similar to Exercise 4.5(b), but **not** *identical to it.*

Problem 5. (12%) Let X and Y be independent with densities

$$f_X(x) = e^{-x}\,,\ x \geq 0 \qquad \text{and} \qquad f_Y(y) = \frac{e^{-1/y}}{y^2}\,,\ y \geq 0.$$

Find the density of XY.

This is similar to Exercise 4.23.

Solution: Make a bivariate transformation $U = XY, V = Y$. Find the joint density of (U, V), and then integrate out V to find the marginal density for $U = XY$.

The inverse transformation is $X = U/V$, $Y = V$ which has Jacobian $J = 1/V$. The support of (U, V) is the first quadrant: $U > 0, V > 0$.

The joint density of (U, V) is

$$f_{U,V}(u, v) = f_{X,Y}(u/v, v) \cdot \frac{1}{v} = e^{-u/v} \cdot \frac{e^{-1/v}}{v^2} \cdot \frac{1}{v} = \frac{e^{-(u+1)/v}}{v^3} \qquad \textit{for } u > 0, v > 0.$$

The density of $U = XY$ is then

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u, v)\,dv = \int_0^{\infty} \frac{e^{-(u+1)/v}}{v^3}\,dv \qquad (\dagger)$$

(Make the substitution $v = 1/s$, $dv = -ds/s^2$.)

$$= \int_{\infty}^{0} s^3 e^{-(u+1)s}\,(-ds/s^2) = \int_0^{\infty} s e^{-(u+1)s}\,ds = \frac{1}{(u+1)^2} \int_0^{\infty} (u+1) s e^{-(u+1)s}\,(u+1)ds$$

$$= \frac{1}{(u+1)^2} \cdot \int_0^{\infty} t e^{-t}\,dt = \frac{1}{(u+1)^2}$$

Thus, the density of $U = XY$ is $f_U(u) = \dfrac{1}{(u+1)^2}$ for $u > 0$.

There are, of course, other ways to carry out the integration.
If students do everything except they are not able to evaluate the integral (†), they should just lose a few points (maybe 2). But they should at least set up the integral (†) correctly (or they should lose more points).

Problem 6. (5%) Here is a joint density function:

$$f_{X,Y}(x, y) = \frac{1}{2\pi(1 - e^{-8})}\, e^{-x^2/2} \sin^2(y) e^{-y^2/8}\,, \quad -\infty < x < \infty\,,\ -\infty < y < \infty\,.$$

Find the marginal density for X. (Simplify as much as possible.)

The marginal density is $\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$*, the standard normal density.*

We may easily see this using the Lemma in the discussion on pages 10 and 11 of notes10.pdf. The joint density has the form

$$\begin{aligned} f(x, y) &= cp(x)q(y) \quad \text{for } -\infty < x < \infty,\ -\infty < y < \infty \\ \text{where } c &= \frac{1}{2\pi(1 - e^{-8})},\ \ p(x) = e^{-x^2/2},\ \ q(y) = \sin^2(y) e^{-y^2/8} \end{aligned}$$

so that we know immediately that the marginal density of X *is* $p(x)$ *up to a normalizing constant. Since we recognize* $p(x)$ *as the kernel of the* $N(0, 1)$ *density, we know the normalizing constant is* $1/\sqrt{2\pi}$ *and we get the answer stated earlier.*

The Lemma is very easy to prove and many students might see on their own (without explicitly using the lemma) that the marginal density must have the form constant $\times\, e^{-x^2/2}$ *and therefore must be the* $N(0, 1)$ *density. Give students full credit if they give any argument like this.*

If a student finds the marginal by brute force integration over y *in the joint density, they should show enough detail so you can see they are actually doing it themselves (and not using a fancy calculator which can do calculus, which they are not supposed to be using on the exam).*

The remaining problems require no work. You will receive full credit for stating the correct answer.

Problem 7. Suppose $X_0, X_1, X_2, \ldots$ is a Markov chain with state space $S = \{1, 2, \ldots, m\}$, initial probability distribution given by the row vector $\boldsymbol{a}$ with entries $a_i = P(X_0 = i)$, and transition probability matrix $\boldsymbol{P}$.

(a) (5%) Give a general expression for $P(X_n = k)$. (Use matrix notation and state your answer as a specific entry in a specific vector or matrix for which you give a formula.)

The answer is $(\boldsymbol{a}\boldsymbol{P}^n)_k$*. See page 14 of notes13_markov_chains.pdf.*

$P(X_n = k) =$ ________________________

(b) (5%) Suppose there are 4 states ($m = 4$) and that the chain is irreducible and aperiodic with stationary distribution $\boldsymbol{\pi} = (0.4, 0.3, 0.2, 0.1)$. Suppose we compute the matrix product $\boldsymbol{P}^h$ where $\boldsymbol{P}$ is the transition probability matrix and h is a huge number (say, $h = 10^{100}$). Then $\boldsymbol{P}^h$ will be approximately equal to what matrix? (Write this matrix explicitly in the space provided.)

The answer is $\begin{pmatrix} .4 & .3 & .2 & .1 \\ .4 & .3 & .2 & .1 \\ .4 & .3 & .2 & .1 \\ .4 & .3 & .2 & .1 \end{pmatrix}$.

$\boldsymbol{P}^h \approx$ ________________________

Problem 8. (3%) Simplify the expression given below. Write your answer in the space provided:

$$\frac{d}{dx}\int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} g(u,y)\,dy\right)\,du \;=\; \underline{\hspace{6cm}}$$

(Assume the function g is such that the integrals are finite and the derivative exists and all necessary regularity conditions are true.)

The answer is $\int_{-\infty}^{\infty} g(x,y)\,dy$. *See the discussion on page 10 of notes9.pdf.*

Problem 9. (3%) For a Markov process,

$$P(X_{n+1} = i_{n+1}\,|\,X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0) = \ldots$$

Circle the response which correctly completes this equation.

a) $P(X_{n+1} = i_1\,|\,X_n = i_0)$ **b**)$\star$ $P(X_{n+1} = i_{n+1}\,|\,X_n = i_n)$ **c**) $P(X_1 = i_1\,|\,X_0 = i_0)$

d) $P(X_{n+1} = i_{n+1}\,|\,X_0 = i_0)$ **e**) $P(X_{n+1} = i_{n+1}\,|\,X_0 = i_n)$ **f**) $P(X_0 = i_n\,|\,X_1 = i_{n+1})$

See page 3 of notes13_markov_chains.pdf