TEST #1
STA 5326 **Name:**__________________________________
September 28, 2018

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems except those on the last page. **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **7** problems and **9** pages. There are a total of **100** points.

Problem 1. Suppose A, B, C are arbitrary events. Derive general expressions for the probabilities in parts (a) and (b) below. Your answers **must** be written as some combination of the values $P(A), P(B), P(C), P(A \cap B), P(A \cap C), P(B \cap C), P(A \cap B \cap C)$.

(a) (16%) $P(A \cap B^c \cap C^c)$

This is exercise B4. See the solution in mordor. This problem can also be done by the method on page 12 of notes4.pdf. (This method is in the Test #2 material, but we have already covered it in lecture, so it is OK if students use it.)

[**Problem 1 continued**]

(b) (8%) $P((A \cup B) \cap C^c)$

This combines arguments from the solution of exercise B4 and the proof of the Principle of Inclusion-Exclusion given on page 10 of notes1.pdf.

$$
\begin{aligned}
P((A \cup B) \cap C^c) &= P((A \cap C^c) \cup (B \cap C^c)) \\
&\quad \textit{(since intersection distributes over union)} \\
&= P(A \cap C^c) + P(B \cap C^c) - P((A \cap C^c) \cap (B \cap C^c)) \\
&\quad \textit{(since } P(A \cup B) = P(A) + P(B) - P(A \cap B) \textit{ for all events } A, B\textit{)} \\
&= [P(A) - P(A \cap C)] + [P(B) - P(B \cap C)] - P(A \cap B \cap C^c) \\
&= P(A) + P(B) - P(A \cap C) - P(B \cap C) - [P(A \cap B) - P(A \cap B \cap C)] \\
&= P(A) + P(B) - P(A \cap C) - P(B \cap C) - P(A \cap B) + P(A \cap B \cap C)
\end{aligned}
$$

This can also be done in a different order as follows:

$$
\begin{aligned}
P((A \cup B) \cap C^c) &= P(A \cup B) - P((A \cup B) \cap C) \\
&\quad \textit{(since } P(A \cap B^c) = P(A) - P(A \cap B) \textit{ for all } A, B\textit{)} \\
&= [P(A) + P(B) - P(A \cap B)] - P((A \cap C) \cup (B \cap C)) \\
&= P(A) + P(B) - P(A \cap B) - [P(A \cap C) + P(B \cap C) - P((A \cap C) \cap (B \cap C))] \\
&= P(A) + P(B) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)
\end{aligned}
$$

Problem 2. (16%) Suppose U has a Uniform$(0,1)$ distribution. Define the random variable Y by

$$Y = \begin{cases} 2 & \text{for } U < 1/3 \\ 3U + 3 & \text{for } U \geq 1/3\,. \end{cases}$$

Find $F_Y(y)$, the cumulative distribution function (cdf) of Y.

This is similar to the example on page 6 of notes3.pdf, and also similar to exercise 1.55.

Let $g(U)$ be the function defined above so that $Y = g(U)$. (It helps to draw a picture of g.) The rv U takes values in $(0,1)$. The function g maps $(0,1)$ into the set $\mathcal{Y} = \{2\} \cup [4,6)$. Thus Y has a lower bound of 2 and an upper bound of 6, and never takes values between 2 and 4. Therefore we know that $F_Y(y) = 0$ for $y < 2$ and $F_Y(y) = 1$ for $y \geq 6$, and that $F_Y(y)$ is flat in the interval $(2,4)$. The entire interval $(0,1/3)$ is mapped into $\{2\}$ by g. Thus $P(0 < U < 1/3) = P(Y = 2) = 1/3$ so that the cdf of Y jumps by 1/3 at $y = 2$. For $4 < y < 6$, it is easily seen from the graph of g that $Y \leq y$ iff $U \leq (y-3)/3$ so that $F_Y(y) = F_U((y-3)/3) = (y-3)/3$. Putting this all together we obtain

$$F_Y(y) = \begin{cases} 0 & y < 2 \\ 1/3 & 2 \leq y < 4 \\ (y-3)/3 & 4 \leq y < 6 \\ 1 & y \geq 6\,. \end{cases}$$

Problem 3. (12%) Suppose $U \sim \text{Uniform}(0,1)$ and X is a random variable with density

$$f_X(x) = \frac{x}{2} \quad \text{for } 0 < x < 2.$$

Find a monotone function $g(\cdot)$ such that $g(U) \stackrel{d}{=} X$.

(You must give an explicit formula for g.)

This exercise uses the result on page 19 of notes3.pdf:

$$\textit{If } X \textit{ has a continuous cdf } F, \textit{ then } F^{-1}(U) \stackrel{d}{=} X.$$

This fact implies that $g = F^{-1}$ is the desired function. To complete the solution, we need to calculate $F = F_X$, and then find F^{-1}.

Integrating the density $f_X(x)$, we find that $F(x) = F_X(x) = x^2/4$ for $0 < x < 2$. Solving $y = F(x) = x^2/4$ for x then gives, $x = F^{-1}(y) = \sqrt{4y}$ for $0 < y < 1$. We conclude that $g(u) = \sqrt{4u}$ is the desired function.

Problem 4. An old man with a very poor memory has 13 friends, all with different ages. Every day he calls one of his friends at random with all 13 friends being equally likely to be chosen on any day. (Because of his poor memory he could even call the same friend two or more days in a row.)

(a) (8%) What is the probability that in the next six days he will call six different friends? (That is, he never calls the same friend more than once.)

This problem is similar to part of exercise B5: finding the probability a poker hand contains nothing. In particular, it is similar to finding the probability that a poker hand contains no repeated values. This problem can be solved in (at least) two different ways.

A counting solution: The outcomes ω of this experiment are 6-tuples: $\omega = (f_1, f_2, \ldots, f_6)$ where f_i denotes the friend called on day i. There are 13^6 equally likely ways ω in which the old man could choose friends to call in the next 6 days: $\#(\Omega) = 13^6$. Let A be the event that the 6 friends are all different. An outcome ω in which the 6 friends are all different may be constructed in two steps: (1) choose 6 of the 13 friends, and then (2) assign each friend to a different day. Thus $\#(A) = \binom{13}{6}6!$ and $P(A) = \#(A)/\#(\Omega) = \binom{13}{6}6!/13^6$

A probability solution: Let A_i be the event that the friend called on day i is different from all the previously called friends. Then $A = A_2 \cap A_3 \cap \cdots \cap A_6$ and

$$\begin{aligned} P(A) &= P(A_2)P(A_3|A_2)P(A_4|A_2 \cap A_3)P(A_5|A_2 \cap A_3 \cap A_4)P(A_6|A_2 \cap A_3 \cap A_4 \cap A_5) \\ &= \frac{12}{13} \cdot \frac{11}{13} \cdot \frac{10}{13} \cdot \frac{9}{13} \cdot \frac{8}{13} \end{aligned}$$

which is the same as the earlier answer.

(b) (8%) What is the probability that the friends he calls in the next six days will have a strictly increasing sequence of ages? In symbols, if we let A_i be the age of the friend he calls on day i, what is the probability that $A_1 < A_2 < A_3 < A_4 < A_5 < A_6$?

This problem is a disguised version of exercise B2(a). So see the solution to B2(a). If you assign the 13 letters $A, B, C, \ldots, K, L, M$ to the 13 friends in order from youngest to oldest, then $A_1 < A_2 < A_3 < A_4 < A_5 < A_6$ is the same as saying the friends are distinct (no repeats) and called in alphabetical order.

The answer is $\binom{13}{6}/13^6$.

[Problem 4 continued]

(c) (8%) What is the probability that $A_1 \le A_2 \le A_3 \le A_4 \le A_5 \le A_6$?

(Note that $A_1 = A_2$ can only happen if he calls the same friend on days 1 and 2 because all the friends have different ages.)

This problem is a disguised version of exercise B2(b). So see the solution to B2(b). If you assign letters $A, B, \ldots, M$ to the 13 friends from youngest to oldest, then $A_1 \le A_2 \le A_3 \le A_4 \le A_5 \le A_6$ just says the friends called are in alphabetical order.

The answer is $\binom{6+13-1}{6}/13^6 = \binom{18}{6}/13^6$. The numerator is the number of 13-tuples of non-negative integers with sum 6; each such 13-tuple corresponds to one outcome $\omega = (f_1, f_2, f_3, f_4, f_5, f_6)$ with the friends satisfying $A_1 \le A_2 \le A_3 \le A_4 \le A_5 \le A_6$.

[**Problem 4 continued**]

(d) (8%) What is the probability that in the next six days there will be somebody that he calls **at least three days in a row?**

("Three days in a row" means the same as "three consecutive days".)

This problem is similar to the "Monkey types AAA" problem, see page 13 of notes1.pdf. The solution uses the principle of inclusion-exclusion.

Let A be the event of interest, that there is someone who is called at least three days in a row. We will use the following notation for events: B_{ijk} is the event that the same friend is called on days i, j, and k; similarly B_{ijkl} is the event that the same friend is called on days i, j, k, and l, etc. Then

$$A = B_{123} \cup B_{234} \cup B_{345} \cup B_{456}$$

so that (by inclusion-exclusion)

$$\begin{aligned}
P(A) &= P(B_{123}) + P(B_{234}) + P(B_{345}) + P(B_{456}) \\
&\quad - P(B_{123} \cap B_{234}) - P(B_{123} \cap B_{345}) - P(B_{123} \cap B_{456}) \\
&\quad - P(B_{234} \cap B_{345}) - P(B_{234} \cap B_{456}) - P(B_{345} \cap B_{456}) \\
&\quad + P(B_{123} \cap B_{234} \cap B_{345}) + P(B_{123} \cap B_{234} \cap B_{456}) \\
&\quad + P(B_{123} \cap B_{345} \cap B_{456}) + P(B_{234} \cap B_{345} \cap B_{456}) \\
&\quad - P(B_{123} \cap B_{234} \cap B_{345} \cap B_{456}) \\
&= P(B_{123}) + P(B_{234}) + P(B_{345}) + P(B_{456}) \\
&\quad - P(B_{1234}) - P(B_{12345}) - P(B_{123})P(B_{456}) - P(B_{2345}) - P(B_{23456}) - P(B_{3456}) \\
&\quad + P(B_{12345}) + P(B_{123456}) + P(B_{123456}) + P(B_{23456}) \\
&\quad - P(B_{123456}) \\
&= 4P(B_{123}) - 3P(B_{1234}) - P(B_{123})P(B_{456}) + P(B_{123456}) \\
&= 4(1/13)^2 - 3(1/13)^3 - (1/13)^2(1/13)^2 + (1/13)^5 \\
&= 4(1/13)^2 - 3(1/13)^3 - (1/13)^4 + (1/13)^5\,.
\end{aligned}$$

Note that $P(B_{ijk}) = (1/13)^2$, $P(B_{ijkl}) = (1/13)^3$, etc. since the friend on day i can be any of the 13 friends.

No work is required for the problems on this page. You will receive full credit just for giving the correct answer.

Problem 5. (8%) Let $A_1, A_2, \ldots, A_n$ be arbitrary sets. Use DeMorgan's Laws to fill in the two blanks below with the expressions which correctly complete these equations.

$$\bigcup_{i=1}^{n} A_i^{\mathbf{c}} = \underline{\left(\bigcap_{i=1}^{n} A_i\right)^c} \qquad\qquad \left(\bigcup_{i=1}^{n} A_i\right)^{\mathbf{c}} = \underline{\bigcap_{i=1}^{n} A_i^{\mathbf{c}}}$$

Problem 6. (4%) If $P(A^{\mathbf{c}}) = 8/13$ and $P(B^{\mathbf{c}}) = 3/13$, then which **one** of the following is possible? (Circle the single correct response.)

a) $A \cap B = \emptyset$ **b**)$\star$ $A \subset B$ **c**) $B \subset A$

Problem 7. (4%) In lecture we gave a formula for the density of $Y = g(X)$ where g is a smooth non-monotonic function and X has a density f_X. This formula is partly given below:

$$f_Y(y) = \sum_{x \,\in\, g^{-1}(\{y\})} \boxed{} \quad \text{for } y \in \mathcal{Y}$$

What goes inside the box? Put your answer below.

See page 24 of notes3.pdf. The complete formula is:

$$f_Y(y) = \sum_{x \in g^{-1}(\{y\})} f_X(x) \left| \frac{1}{g'(x)} \right| \quad \textit{for } y \in \mathcal{Y}$$

So what goes inside the box is $f_X(x) \left| \frac{1}{g'(x)} \right|$. *There is another formula for* $f_Y(y)$ *which involves terms like* $f_X(g_i^{-1}(y)) |\frac{d}{dy} g_i^{-1}(y)|$, *but this is the wrong response to this problem because it does not involve summing over* $x \in g^{(-1)}(\{y\})$.