TEST #3
STA 5326 **Name:**____________________________________
December 6, 2018

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems except those on the last three pages. **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **7** problems and **11** pages. There are a total of **100** points.

Problem 1. (8%) Prove the following:

$$\text{If } a \leq g(x,y) \leq b \text{ for all } x,y, \text{ then } a \leq Eg(X,Y) \leq b.$$

You may assume that X and Y have a joint density $f(x,y)$.

Problem 2. Suppose

$$Y|X \sim \text{NegativeBinomial}(r, X) \quad \text{and} \quad X \sim \text{Beta}(\alpha, \beta)\,.$$

(This problem uses the textbook's definition of the NegativeBinomial distribution, which is the main definition given in the Table of Common Distributions.)

(a) (12%) Find the marginal pmf of Y.

[**Problem 2 continued**]

(b) (12%) Find EY.

[**Problem 2 continued**]

(c) (12%) Find $\mathrm{Var}(Y)$.

Problem 3. (10%) Let X and Y be independent NegativeBinomial$(1, p)$ random variables. Find the distribution of $X|X+Y$. That is, for any integers j and k with $0 \leq j \leq k$, find $P(X = j \,|\, X + Y = k)$.

(Again, this problem uses the textbook's definition of the NegativeBinomial distribution. This distribution satisfies a closure property which tells us that $X + Y \sim$ NegativeBinomial$(2, p)$. You may use this fact without proof in your solution.)

Problem 4. Let X and Y be independent $N(0, \sigma^2)$ random variables.

(a) (20%) Find the joint density of U and W defined by

$$U = \sqrt{X^2 + Y^2} \quad \text{and} \quad W = \frac{Y}{\sqrt{X^2 + Y^2}} \, .$$

Your answer should clearly specify the support of the joint density.

[**Problem 4 continued**]

(b) (4%) Are U and W independent? Answer "Yes" or "No" and prove your answer.

The remaining problems require no work to be shown. You will receive full credit just for stating the correct answer.

Problem 5. (6%) Suppose X and Y are random variables which each take values in $\{1, 2, 3, 4\}$. Suppose you have carried out calculations using the joint mass function of X and Y and obtained the information in the table below:

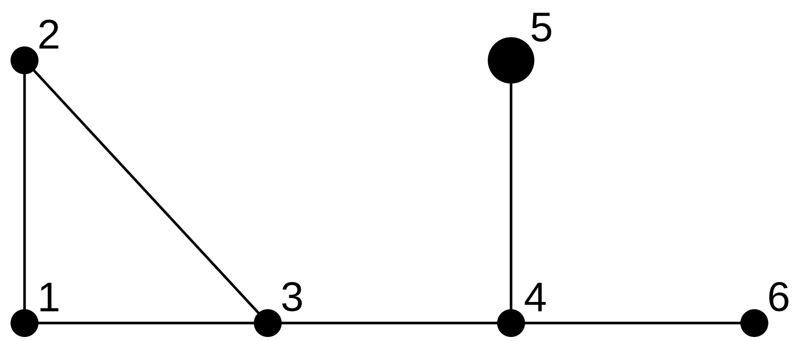
X	$f_X(X)$	$E(Y\|X)$	$\text{Var}(Y\|X)$	Y	$f_Y(Y)$	$E(X\|Y)$	$\text{Var}(X\|Y)$
1	0.3	2.1666667	0.4722222	1	0.3	2.8333333	1.4722222
2	0.3	3	2	2	0.2	1.75	1.6875
3	0.15	3.3333333	0.2222222	3	0.2	2	1
4	0.25	1.8	1.36	4	0.3	2.5	0.5833333

Using the information in this table, show how to compute $E[\text{Var}(Y|X)]$. Your answer should be a numerical expression and only numbers from the table above should appear in it. You do NOT need to evaluate this expression.

$E[\text{Var}(Y|X)] =$ ____

Problem 6. (8%) The graph below has 6 nodes numbered 1 to 6. Two nodes are called "neighbors" if they are joined by an edge. (For example, nodes 3 and 4 are neighbors, but notes 3 and 6 are **not**.) A sleepwalker is wandering about on the nodes of this graph. He starts at node **2**. At each time $t = 1, 2, 3, \ldots$, he moves according to the rule: with probability 1/2 he stays at the same node; otherwise he chooses from the available neighboring nodes at random (equally likely) and goes to that node. However, node **5** (the BIG node) is absorbing. If he goes to node **5**, he must remain there forever.

The sleepwalker moves according to a Markov chain. What are the initial distribution (the row vector) and transition probability matrix (the matrix $\mathbf{P}$) for this chain?



.

$\mathbf{a} =$ ____

$\mathbf{P} =$ ____

Problem 7. (8%) The initial distribution $\mathbf{a}$ and transition probability matrix P for a Markov chain $X_0, X_1, X_2, \ldots$ with state space $\{1, 2, 3, 4\}$ is given below along with the matrix products P^2 and P^3.

$$\mathbf{a} = \begin{pmatrix} 0.27 & 0.24 & 0.43 & 0.06 \end{pmatrix}$$
$$P = \begin{pmatrix} 0.02 & 0.44 & 0.49 & 0.05 \\ 0.69 & 0.14 & 0.09 & 0.08 \\ 0.38 & 0.29 & 0.11 & 0.22 \\ 0.73 & 0.04 & 0.16 & 0.07 \end{pmatrix}$$
$$P^2 = \begin{pmatrix} 0.5267 & 0.2145 & 0.1113 & 0.1475 \\ 0.203 & 0.3525 & 0.3734 & 0.0711 \\ 0.4101 & 0.2485 & 0.2596 & 0.0818 \\ 0.1541 & 0.376 & 0.3901 & 0.0798 \end{pmatrix}$$
$$P^3 = \begin{pmatrix} 0.3085 & 0.3 & 0.3132 & 0.0783 \\ 0.4411 & 0.2498 & 0.1836 & 0.1255 \\ 0.338 & 0.2938 & 0.265 & 0.1032 \\ 0.469 & 0.2368 & 0.165 & 0.1292 \end{pmatrix}$$

Write $P(X_0 = 4, X_3 = 2, X_5 = 3, X_6 = 1)$ as a product of some of the numerical values given above. (**Write out all the factors explicitly, but do NOT evaluate the product!**)

$P(X_0 = 4, X_3 = 2, X_5 = 3, X_6 = 1) =$ ________________________________