TEST #3
STA 5326 **Name:**__________________________________
December 6, 2018

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems except those on the last three pages. **No credit is given without work.** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **7** problems and **11** pages. There are a total of **100** points.

General Remark: I do NOT require students to simplify their arithmetic. They can leave in fractions, powers, factorials, etc. If you can see that their answer is correct, give them full credit. But students **must** *do all the necessary algebra and calculus to get full credit. They must compute all derivatives; they should lose points if they leave 'd/dx's or 'primes' in their answer. Also, student must simplify summations if there is a simple closed form.*

I usually do not deduct points for arithmetic errors unless they make the answer ridiculous. (For example, probabilities not between 0 and 1, negative expected values for positive-valued random variables, negative variances, etc.).

I usually do not deduct points for copying errors (unless they make the answer ridiculous).

Problem 1. (8%) Prove the following:

$$\text{If } a \le g(x,y) \le b \text{ for all } x, y, \text{ then } a \le Eg(X,Y) \le b.$$

You may assume that X and Y have a joint density $f(x,y)$.

This is part of exercise 4.2. There is no posted solution to this particular problem.

Students can use any standard property of integrals in their solution. For example, the following is an adequate solution:

$$\begin{aligned}
& a \le g(x,y) \le b \textit{ for all } x, y \\
\textit{implies} \quad & af(x,y) \le g(x,y)f(x,y) \le bf(x,y) \textit{ for all } x, y \\
\textit{implies} \quad & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} af(x,y)\,dx\,dy \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dx\,dy \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} bf(x,y)\,dx\,dy \\
\textit{implies} \quad & a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dx\,dy \le b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy \\
\textit{implies} \quad & a \le Eg(X,Y) \le b \qquad \textit{since} \int\int f(x,y)\,dx\,dy = 1.
\end{aligned}$$

Actually, this is more detail than is really needed. A somewhat less detailed solution could still get full credit.

Problem 2. Suppose

$$Y|X \sim \text{NegativeBinomial}(r, X) \quad \text{and} \quad X \sim \text{Beta}(\alpha, \beta)\,.$$

(This problem uses the textbook's definition of the NegativeBinomial distribution, which is the main definition given in the Table of Common Distributions.)

This is exercise 4.34(b).

Students should make the more obvious simplifications, but it is OK if students leave things in terms of Gamma functions in their answers. Also, algebra does not have to be in the most simple form. (This applies to all the problems.)

(a) (12%) Find the marginal pmf of Y.

[**Problem 2 continued**]

(b) (12%) Find EY.

[**Problem 2 continued**]

(c) (12%) Find $\mathrm{Var}(Y)$.

Problem 3. (10%) Let X and Y be independent NegativeBinomial$(1, p)$ random variables. Find the distribution of $X|X + Y$. That is, for any integers j and k with $0 \leq j \leq k$, find $P(X = j \,|\, X + Y = k)$.

(Again, this problem uses the textbook's definition of the NegativeBinomial distribution. This distribution satisfies a closure property which tells us that $X + Y \sim$ NegativeBinomial$(2, p)$. You may use this fact without proof in your solution.)

This is similar to exercise 4.15.

The answer is $P(X = j \,|\, X + Y = k) = 1/(k + 1)$ *for* $0 \leq j \leq k$.

This is a discrete uniform distribution on the set $\{0, 1, \ldots, k\}$ *(but students do NOT have to say this).*

Problem 4. Let X and Y be independent $N(0, \sigma^2)$ random variables.

(a) (20%) Find the joint density of U and W defined by

$$U = \sqrt{X^2 + Y^2} \quad \text{and} \quad W = \frac{Y}{\sqrt{X^2 + Y^2}}\,.$$

Your answer should clearly specify the support of the joint density.

This is very similar to exercise 4.20.

[**Problem 4 continued**]

(b) (4%) Are U and W independent? Answer "Yes" or "No" and prove your answer.

The answer is "Yes" since the support is a Cartesian product set and the density factors as "constant" times "function of U*" times "function of* W*" on the support. You could also just say: independent because the density factors as a "constant" times "function of* U*" times "function of* W*' valid for all* U *and* W*.*

If students get the wrong joint density, but give the correct answer for whatever density they get and give the correct reasoning, you can give them credit.

The remaining problems require no work to be shown. You will receive full credit just for stating the correct answer.

Problem 5. (6%) Suppose X and Y are random variables which each take values in $\{1, 2, 3, 4\}$. Suppose you have carried out calculations using the joint mass function of X and Y and obtained the information in the table below:

X	$f_X(X)$	$E(Y\|X)$	$\text{Var}(Y\|X)$	Y	$f_Y(Y)$	$E(X\|Y)$	$\text{Var}(X\|Y)$
1	0.3	2.1666667	0.4722222	1	0.3	2.8333333	1.4722222
2	0.3	3	2	2	0.2	1.75	1.6875
3	0.15	3.3333333	0.2222222	3	0.2	2	1
4	0.25	1.8	1.36	4	0.3	2.5	0.5833333

Using the information in this table, show how to compute $E[\text{Var}(Y|X)]$. Your answer should be a numerical expression and only numbers from the table above should appear in it. You do NOT need to evaluate this expression.

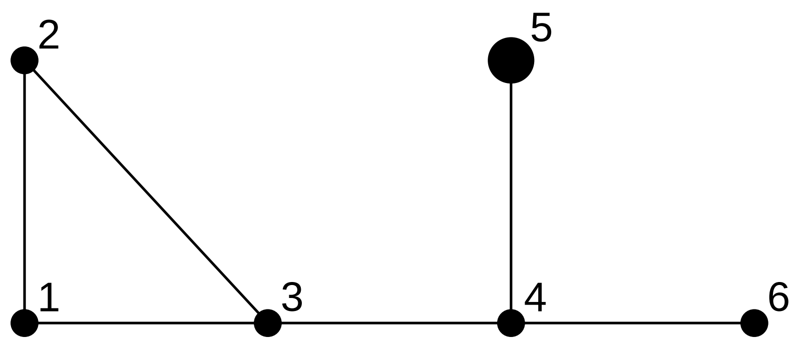
$E[Var(Y|X)]$ is obtained by multiplying the column $f_X(X)$ times the column $Var(Y|X)$ and summing the products.

Students don't need to put their answers in the blanks (as long as it is clear what their answer is).

$E[\text{Var}(Y|X)] =$ ____________________

Problem 6. (8%) The graph below has 6 nodes numbered 1 to 6. Two nodes are called "neighbors" if they are joined by an edge. (For example, nodes 3 and 4 are neighbors, but notes 3 and 6 are **not**.) A sleepwalker is wandering about on the nodes of this graph. He starts at node **2**. At each time $t = 1, 2, 3, \ldots$, he moves according to the rule: with probability 1/2 he stays at the same node; otherwise he chooses from the available neighboring nodes at random (equally likely) and goes to that node. However, node **5** (the BIG node) is absorbing. If he goes to node **5**, he must remain there forever.

The sleepwalker moves according to a Markov chain. What are the initial distribution (the row vector) and transition probability matrix (the matrix $\mathbf{P}$) for this chain?



This is very similar to the sleepwalker example on pages 5 and 6 of notes13_markov_chains.pdf.

You could break down the credit as 2 points for $\mathbf{a}$ *and 6 points for* $\mathbf{P}$. *Any method of assigning partial credit is OK so long as it is consistently applied.*

$\mathbf{a} =$ ____

$\mathbf{P} =$ ____

Problem 7. (8%) The initial distribution $\mathbf{a}$ and transition probability matrix P for a Markov chain $X_0, X_1, X_2, \ldots$ with state space $\{1, 2, 3, 4\}$ is given below along with the matrix products P^2 and P^3.

$$\mathbf{a} = \begin{pmatrix} 0.27 & 0.24 & 0.43 & 0.06 \end{pmatrix}$$
$$P = \begin{pmatrix} 0.02 & 0.44 & 0.49 & 0.05 \\ 0.69 & 0.14 & 0.09 & 0.08 \\ 0.38 & 0.29 & 0.11 & 0.22 \\ 0.73 & 0.04 & 0.16 & 0.07 \end{pmatrix}$$
$$P^2 = \begin{pmatrix} 0.5267 & 0.2145 & 0.1113 & 0.1475 \\ 0.203 & 0.3525 & 0.3734 & 0.0711 \\ 0.4101 & 0.2485 & 0.2596 & 0.0818 \\ 0.1541 & 0.376 & 0.3901 & 0.0798 \end{pmatrix}$$
$$P^3 = \begin{pmatrix} 0.3085 & 0.3 & 0.3132 & 0.0783 \\ 0.4411 & 0.2498 & 0.1836 & 0.1255 \\ 0.338 & 0.2938 & 0.265 & 0.1032 \\ 0.469 & 0.2368 & 0.165 & 0.1292 \end{pmatrix}$$

Write $P(X_0 = 4, X_3 = 2, X_5 = 3, X_6 = 1)$ as a product of some of the numerical values given above. (**Write out all the factors explicitly, but do NOT evaluate the product!**)

The answer is a product of four numbers. You could break down the credit as 2 points per number (or any other way that seems reasonable to you).

$P(X_0 = 4, X_3 = 2, X_5 = 3, X_6 = 1) =$ ____________________________________