TEST #1
STA 5326 **Name:**_____
October 1, 2019

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems except those on the last two pages. **No credit is given without work or explanation!** This even includes the counting problems. But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion. (Exception: there is one multiple choice problem where you are asked to circle **all** correct responses.)
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and **12** pages. There are a total of **100** points.

This problem involves a standard deck of 52 cards. The cards have 13 different values (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K) and the deck has 4 cards of each value. The cards have 4 different suits (hearts, diamonds, spades, clubs) and the deck has 13 cards of each suit.

Problem 1. Suppose that **12** cards are dealt from a well shuffled deck. Answer the following.

(a) (7%) What is the probability these 12 cards contain a pair of 1's, a pair of 2's, a pair of 3's, a pair of 4's, a pair of 5's, and a pair of 6's?

(b) (5%) What is the probability these 12 cards contain **NO** hearts?

(c) (8%) What is the probability the 12 cards consist of 5 pairs and two singletons?

(Note: A "pair" is two cards which have the same value, with this value being different from all the other values in the set of 12 cards. A "singleton" is a single card whose value is different from all the other values in the set of 12 cards.)

Problem 2. Suppose you have a fair coin with the sides labeled $+1$ and -1. Toss this coin 3 times and let X_i be the value observed on the i-th toss. Define $X_4 = X_1X_2X_3$. For $i = 1, 2, 3, 4$, define A_i to be the event that $X_i = 1$.

(a) (9%) Describe (in general) what must be done to show that four events A, B, C, D are mutually independent.

(b) (9%) Show that the events A_1, A_2, A_3, A_4 are *not* mutually independent.

Problem 3. A monkey types **5** letters at random. (Each of the 5 keystrokes is independent of the others with all 26 possibilities equally likely.)

(a) (8%) What is the probability the monkey types $\mathcal{V}\mathcal{V}\mathcal{V}$ where $\mathcal{V}$ stands for any of the 5 vowels A, E, I, O, U?

(Note: "Monkey types $\mathcal{V}\mathcal{V}\mathcal{V}$" means that the monkey types 3 or more consecutive vowels, for example $XXUIA$ or $AAEOZ$.)

(b) (8%) What is the probability the monkey types at least three vowels? (Here they need NOT be consecutive.)

(c) (5%) What is the probability the monkey types $\mathcal{VVV}$ given that the monkey types at least 3 vowels.

Problem 4. (10%) Suppose X has density $f_X(x) = (x+3)/18$ for $-3 < x < 3$ (and 0 otherwise). Let $Y = X(X+1)(X-2)$. Find $f_Y(0)$, the pdf of Y evaluated at 0. (Clearly state any results you are using in your solution.)

Problem 5. Consider the function

$$F(x) = \begin{cases} e^{-e^{-x}} & \text{for } x < 0 \\ 1 - e^{-e^{x}} & \text{for } x \geq 0\,. \end{cases}$$

(Recall that $e \approx 2.718$.)

(a) (1%) Is F a cumulative distribution function (cdf)? (Answer 'Yes' or 'No'.)

(b) (9%) Prove your answer. (Clearly state the requirements for F to be a cdf, and then explicitly verify each of these requirements or show that one of them fails.)

Problem 6. (8%) Suppose Y has cumulative distribution function (cdf) given by

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 1 \\ 1 - e^{1-y} & \text{for } y \geq 1 \end{cases}$$

Find the cdf of $Z = 5(Y - 2)$. Write your final answer as a piecewise expression as was used in the definition of $F_Y(y)$ given above.

In the remaining problems, no work or explanation is required. You will receive full credit just for giving the correct answers. No partial credit is given.

Problem 7. (7%) Suppose that A and B are disjoint and that $P(A) > 0$ and $P(B) > 0$. Under these assumptions, which of the following statements are always true? (Circle **all** the correct responses. No work is required.)

a) $P(B \mid A) = 0$

b) $P(B \mid A^c) = P(B)$

c) $P(A \mid A \cup B) = P(A)/(P(A) + P(B))$

d) $P(A \mid B) = P(A)$

e) $P(A \cap B) = P(A)P(B)$

f) $P(A \cap B \mid A) = 0$

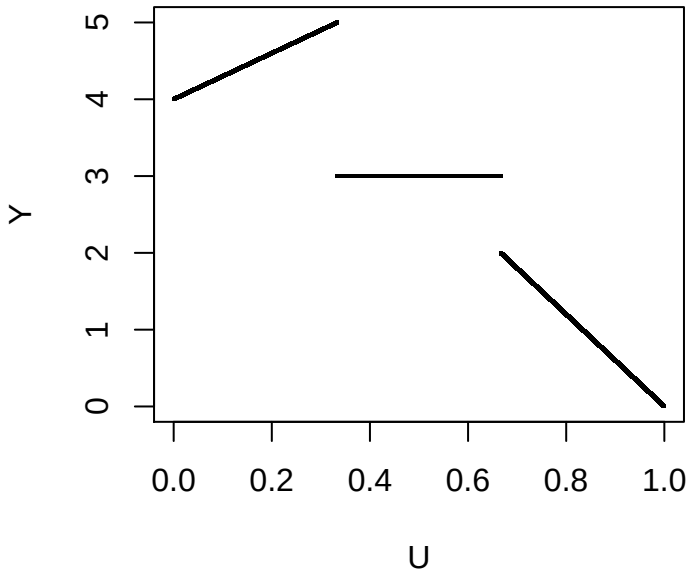
g) $P(B^c \mid A) = 1$

The remaining space on this page is for scratch work. This work will NOT be examined and NO partial credit is given.

Problem 8. (6%) Suppose $U \sim \text{Uniform}(0, 1)$ and

$$Y = \begin{cases} 4 + 3U & \text{for } U < 1/3 \\ 3 & \text{for } 1/3 \leq U < 2/3 \\ 6 - 6U & \text{for } 2/3 \leq U \end{cases}$$

A graph of Y as a function of U is given below.



Find the following. Here $F_Y(\cdot)$ denotes the cdf of Y. (Just fill in the blanks. You are NOT required to find the cdf.)

(a) $P(2 < Y < 3) =$ _____

(b) $F_Y(3) - F_Y(3-) =$ _____

(c) $F_Y(2) - F_Y(0) =$ _____