**TEST #1**
**STA 5326** Name:______________________________________
**October 1, 2019**

**Please read the following directions.**
**DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO**

# Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems except those on the last two pages. **No credit is given without work or explanation!** This even includes the counting problems. But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion. (Exception: there is one multiple choice problem where you are asked to circle **all** correct responses.)
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely. Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **8** problems and **12** pages. There are a total of **100** points.

*General Remarks on grading:*

*I do NOT require students to simplify their arithmetic. They can leave in fractions, powers, factorials, etc. If you can see that their answer is correct, give them full credit. But students* **must** *do all the necessary algebra and calculus to get full credit. They must compute all derivatives; they should lose points if they leave 'd/dx's or 'primes' in their answer. Also, student must simplify summations if there is a simple closed form.*

*I usually do not deduct points for arithmetic errors unless they make the answer ridiculous. (For example, probabilities not between 0 and 1, negative expected values for positive-valued random variables, negative variances, etc.).*

*I usually do not deduct points for copying errors (unless they make the answer ridiculous).*

*I do deduct points for algebra and calculus errors, but as long as the student's solution is basically correct (except for the algebra or calculus mistakes), I try to give them at least half the credit.*

*However, if an algebra or calculus mistake is bad enough to transform the problem so that the solution after the point of the mistake no longer resembles the correct solution, then the student only gets credit for the work up to the point of the mistake.*

*In general, if a student makes a mistake which completely changes the problem, and then the student correctly solves the changed problem, they only get credit up to the point of the mistake; they don't get credit for correctly solving the wrong problem.*

This problem involves a standard deck of 52 cards. The cards have 13 different values (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K) and the deck has 4 cards of each value. The cards have 4 different suits (hearts, diamonds, spades, clubs) and the deck has 13 cards of each suit.

**Problem 1.** Suppose that **12** cards are dealt from a well shuffled deck. Answer the following.

**(a)** (7%) What is the probability these 12 cards contain a pair of 1's, a pair of 2's, a pair of 3's, a pair of 4's, a pair of 5's, and a pair of 6's?

*This is similar to exercise 1.22(a).*

**(b)** (5%) What is the probability these 12 cards contain **NO** hearts?

*This is similar to exercise 1.22(b).*

**(c)** (8%) What is the probability the 12 cards consist of 5 pairs and two singletons?

(Note: A "pair" is two cards which have the same value, with this value being different from all the other values in the set of 12 cards. A "singleton" is a single card whose value is different from all the other values in the set of 12 cards.)

**Problem 2.** Suppose you have a fair coin with the sides labeled $+1$ and $-1$. Toss this coin 3 times and let $X_i$ be the value observed on the $i$-th toss. Define $X_4 = X_1 X_2 X_3$. For $i = 1, 2, 3, 4$, define $A_i$ to be the event that $X_i = 1$.

**(a)** (9%) Describe (in general) what must be done to show that four events $A, B, C, D$ are mutually independent.

*This is exercise B8(a).*

**(b)** (9%) Show that the events $A_1, A_2, A_3, A_4$ are *not* mutually independent.

*This is exercise B8(b).*

**Problem 3.** A monkey types **5** letters at random. (Each of the 5 keystrokes is independent of the others with all 26 possibilities equally likely.)

**(a)** (8%) What is the probability the monkey types $\mathcal{VVV}$ where $\mathcal{V}$ stands for any of the 5 vowels $A, E, I, O, U$?

(Note: "Monkey types $\mathcal{VVV}$" means that the monkey types 3 or more consecutive vowels, for example $XXUIA$ or $AAEOZ$.)

*This is similar to $P$(monkey types $AAA$) example on page 13 of notes1.pdf. The solution is the same except $A$ is everywhere replaced by $\mathcal{V}$ and $1/26$ is replaced by $5/26$. The answer ends up being $3(5/26)^3 - 2(5/26)^4$. (Students should show all the details.)*

**(b)** (8%) What is the probability the monkey types at least three vowels? (Here they need NOT be consecutive.)

*This is similar to exercise 1.36. It is a typical Binomial distribution problem. The number of vowels the monkey types has a Binomial*($n = 5, p = 5/26$) *distribution, so the probability of at least 3 vowels is*

$$\binom{5}{3}(5/26)^3(21/26)^2 + \binom{5}{4}(5/26)^4(21/26) + \binom{5}{5}(5/26)^5 .$$

**(c)** (5%) What is the probability the monkey types $\mathcal{VVV}$ given that the monkey types at least 3 vowels.

*This is somewhat similar to exercise 1.36.*

*Let* $A = \{$*monkey types* $\mathcal{VVV}\}$ *and* $B = \{$*monkey types at least 3 vowels*$\}$. *We are asked to find* $P(A \mid B) = P(A \cap B)/P(B) = P(A)/P(B)$ *since* $A \subset B$. *So the answer is* $P(A)/P(B)$ *where* $P(A)$ *is the answer to part (a) of this problem and* $P(B)$ *is the answer to part (b).*

**Problem 4.** (10%) Suppose $X$ has density $f_X(x) = (x+3)/18$ for $-3 < x < 3$ (and 0 otherwise). Let $Y = X(X+1)(X-2)$. Find $f_Y(0)$, the pdf of $Y$ evaluated at 0. (Clearly state any results you are using in your solution.)

*The easiest way to do this problem is to use the result on page 24 of notes3.pdf:*

$$f_Y(y) = \sum_{x \in g^{-1}(\{y\})} f_X(x) \left| \frac{1}{g'(x)} \right| .$$

*With* $g(x) = x(x+1)(x-2)$, *we have* $g^{-1}(\{0\}) = \{-1, 0, 2\}$ *and*

$$g'(x) = (x+1)(x-2) + x(x-2) + x(x+1) .$$

*So the answer is*

$$\begin{aligned} f_Y(0) &= f_X(-1) \left| \frac{1}{g'(-1)} \right| + f_X(0) \left| \frac{1}{g'(0)} \right| + f_X(2) \left| \frac{1}{g'(2)} \right| \\ &= \frac{2}{18} \cdot \frac{1}{3} + \frac{3}{18} \cdot \frac{1}{2} + \frac{5}{18} \cdot \frac{1}{6} = \frac{1}{6} . \end{aligned}$$

**Problem 5.** Consider the function

$$F(x) = \begin{cases} e^{-e^{-x}} & \text{for } x < 0 \\ 1 - e^{-e^{x}} & \text{for } x \geq 0\,. \end{cases}$$

(Recall that $e \approx 2.718$.)

**(a)** (1%) Is $F$ a cumulative distribution function (cdf)? (Answer 'Yes' or 'No'.)

*Answer: Yes, $F$ is a cdf.*

**(b)** (9%) Prove your answer. (Clearly state the requirements for $F$ to be a cdf, and then explicitly verify each of these requirements or show that one of them fails.)

*We must verify the three conditions on page 22 of notes2.pdf. Students should list these conditions or use them very explicitly in their work.*

*Define $g(x) = e^{-e^{-x}}$ and $h(x) = 1 - e^{-e^{x}}$.*

*We know that $e^{\infty} \equiv \lim_{x\to\infty} e^x = \infty$ and $e^{-\infty} \equiv \lim_{x\to-\infty} e^x = 0$. Thus*

$$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} g(x) = e^{-e^{\infty}} = e^{-\infty} = 0 \quad \textit{and}$$
$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} h(x) = 1 - e^{-e^{\infty}} = 1 - e^{-\infty} = 1 - 0 = 1\,.$$

*We note that $g'(x) = e^{-e^{-x}} \cdot e^{-x} > 0$ so that $g$ is continuous and strictly increasing for all $x$. Similarly, $h'(x) = e^{-e^{x}} \cdot e^{x} > 0$ so that $h$ is continuous and strictly increasing for all $x$. Thus $F$ is strictly increasing and continuous on $(-\infty, 0)$ (where it equals $g$) and on $(0, \infty)$ (where it equals $h$). The only discontinuity of $F$ is at $x = 0$, but there we have $F(0-) = g(0) = 1/e < 1 - (1/e) = h(0) = F(0) = F(0+)$. Since $F$ is right continuous and jumps upward at the point $x = 0$, it is clear that $F$ is nondecreasing and right-continuous for all $x$.*

**Problem 6.** (8%) Suppose $Y$ has cumulative distribution function (cdf) given by

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 1 \\ 1 - e^{1-y} & \text{for } y \geq 1 \end{cases}$$

Find the cdf of $Z = 5(Y - 2)$. Write your final answer as a piecewise expression as was used in the definition of $F_Y(y)$ given above.

*This is similar to exercise 1.53(c).*

*The cdf of $Z$ is*

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(5(Y-2) \leq z) = P(Y \leq (z/5) + 2) \\ &= F_Y((z/5)+2) = \begin{cases} 0 & \textit{for } (z/5)+2 < 1 \\ 1 - \exp(1 - [(z/5)+2]) & \textit{for } (z/5)+2 \geq 1 \end{cases} \\ &= \begin{cases} 0 & \textit{for } z < -5 \\ 1 - \exp(-(z+5)/5) & \textit{for } z \geq -5. \end{cases} \end{aligned}$$

**In the remaining problems, no work or explanation is required. You will receive full credit just for giving the correct answers. No partial credit is given.**

**Problem 7.** (7%) Suppose that $A$ and $B$ are disjoint and that $P(A) > 0$ and $P(B) > 0$. Under these assumptions, which of the following statements are always true? (Circle **all** the correct responses. No work is required.)

**a)**$\star$ $P(B \mid A) = 0$

**b)** $P(B \mid A^c) = P(B)$

**c)**$\star$ $P(A \mid A \cup B) = P(A)/(P(A) + P(B))$

**d)** $P(A \mid B) = P(A)$

**e)** $P(A \cap B) = P(A)P(B)$

**f)**$\star$ $P(A \cap B \mid A) = 0$

**g)**$\star$ $P(B^c \mid A) = 1$

*This question is partly based on exercises 1.38 and 1.39.*

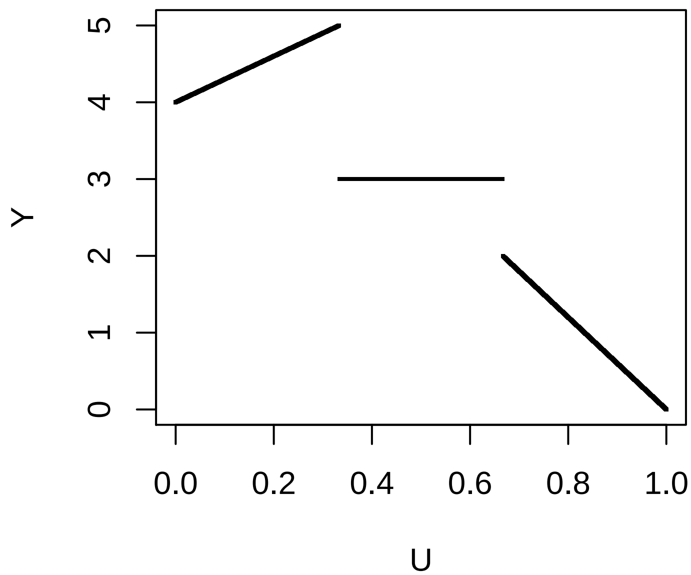
*Give 1 point for each correctly circled and correctly non-circled item. If a student circles only the four correct responses they get 7 points. If they circle only the 3 wrong responses, they get 0 points. This question is just like 7 true/false questions worth 1 point each.*

The remaining space on this page is for scratch work. This work will NOT be examined and NO partial credit is given.

**Problem 8.** (6%) Suppose $U \sim \text{Uniform}(0,1)$ and

$$Y = \begin{cases} 4+3U & \text{for } U < 1/3 \\ 3 & \text{for } 1/3 \le U < 2/3 \\ 6-6U & \text{for } 2/3 \le U \end{cases}$$

A graph of $Y$ as a function of $U$ is given below.




Find the following. Here $F_Y(\cdot)$ denotes the cdf of $Y$. (Just fill in the blanks. You are NOT required to find the cdf.)

*This problem is similar to the earlier parts of the example on pages 5 to 8 of notes3.pdf where certain probabilities and properties of the cdf are determined by inspection of the graph of $Y$ as a function of $U$.*

**(a)** $P(2 < Y < 3) =$ __________________

**(b)** $F_Y(3) - F_Y(3-) =$ __________________

**(c)** $F_Y(2) - F_Y(0) =$ __________________