TEST #2
STA 5326 **Name:**____________________________________
November 7, 2019

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems except those on the last two pages. **No credit is given without work or explanation!** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely (except as noted below). Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- **On this exam, problems 1(a), 1(b), and 5 require decimal answers for full credit.**
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **9** problems and **11** pages. There are a total of **100** points.

Problem 1. Joe really likes Nutro Meal Replacement Bars and eats a lot of them. They are currently holding a sales promotion: each bar has probability $1/2$ of having a coupon wrapped with it. (The bars are independent of each other.) **The different parts below are NOT related.**

(a) (12%) Joe is going to place an order for M Nutro Bars. He would like to have a probability of at least 0.90 of getting at least 100 coupons. What is the smallest value of M which Joe could choose? Use an appropriate approximation. (Your final answer should be an integer. You may wish to use some trial and error to find your answer.)

(b) (8%) Later Joe, hearing that Elizabeth Warren might win the next election and fearing the imminent collapse of civilization, decides to use his entire life savings to order a shipment of 40,000 Nutro Bars. What is the probability that he receives **exactly** 20,000 coupons with his shipment? Use an appropriate approximation. (To receive full credit you must give a decimal answer which does NOT use the normal tables.)

(c) (6%) Suppose that every coupon can be redeemed for a free Nutro Bar, and that each of these free bars also has probability 0.5 of coming with a coupon. Suppose that Joe buys **8** Nutro Bars and always redeems every coupon he receives for a free bar. Let X be the total number of Nutro Bars he ends up with. Give a formula for $f_X(x)$, the probability mass function (pmf) of X.

Problem 2. (12%) Prove the "two-way" rule for expectations (also known as "The Law of the Unconscious Statistician"): If $Y = g(X)$ and $Eg(X)$ exists, then

$$\int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx\,.$$

Assume that g is differentiable and strictly increasing.

Problem 3. Let X have the density (pdf)

$$f(x) = \frac{2}{\beta^2} x e^{-x^2/\beta^2}, \quad 0 < x < \infty \quad (\text{where } \beta > 0).$$

(a) (10%) Find EX.

(b) (10%) Find $\mathrm{Var}(X)$.

Problem 4. (12%) A *truncated* discrete distribution is one in which a particular class cannot be observed and is eliminated from the sample space. In particular, if X has range 0, 1, 2, ... and the 0 class cannot be observed, the 0-*truncated* random variable X_T has mass function (pmf)

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)}, \quad x = 1, 2, 3, \ldots$$

Find the pmf, mean, and variance of the 0-*truncated* random variable starting from $X \sim \text{Poisson}(\lambda)$. (Circle these quantities in your solution.)

Problem 5. (12%) There are 10,000 motorcyclists in a city. The probability that an individual motorcyclist will suffer a serious or fatal accident in a one year period depends on their age as described in the table below. The table also lists the number of motorcyclists in each age category.

Group	# in Group	Prob. of Serious or Fatal Accident
Age $<$ 21 years	2,000	8×10^{-4}
Age between 21 and 50	5,000	2×10^{-4}
Age $>$ 50 years	3,000	4×10^{-4}

Assuming the motorcyclists are independent, what is the probability of exactly 5 serious or fatal accidents in the next year? (Use an appropriate approximation.)

In the remaining problems you will receive full credit just for stating the correct answer. You are not required to show work or give explanation.

Problem 6. (5%) Let $X_1, X_2, X_3, \dots, X_n$ be iid exponential random variables with mean $1/\lambda$. For $t > 0$, what is the value of $P(X_1 + X_2 + \cdots + X_n > t)$? (For full credit, your answer should be a **summation** in which the terms are given explicitly.)

Problem 7. (5%) Let $X \sim \text{Poisson}(\lambda = 13)$. What is the moment generating function (mgf) of $Y = \dfrac{X - 5}{19}$. (You may use the appendix.)

Problem 8. (4%) Let $X_1, X_2, X_3, \ldots, X_n$ be iid exponential random variables with mean β. Define $Y = \min X_i$. For $t > 0$, what is $h_Y(t)$, the hazard function of Y? (Give a simple expression.)

Problem 9. (4%) Suppose $X_1, X_2, X_3, \ldots$ are iid with mean μ and variance $\sigma^2 < \infty$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Which **one** of the following responses is a correct statement of the **Strong** Law of Large Numbers (**S**LLN)?

a) $\lim_{n\to\infty} P\left(\bar{X}_n = \mu\right) = 1$

b) $\lim_{n\to\infty} P\left(|\bar{X}_n - \mu| < \varepsilon\right) = 1$ for all $\varepsilon > 0$

c) $\lim_{n\to\infty} P\left(|\bar{X}_n - \mu| = 0\right) = 1$

d) $P\left(\lim_{n\to\infty} |\bar{X}_n - \mu| > \varepsilon\right) = 1$ for all $\varepsilon > 0$

e) $P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1$

f) $\lim_{n\to\infty} P\left(|\bar{X}_n - \mu| > \varepsilon\right) = 1$ for all $\varepsilon > 0$