Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.

- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).

- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.

- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)

- You must show and explain your work (including your calculations) for all the problems except those on the last two pages. **No credit is given without work or explanation!** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.

- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.

- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.

- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely (except as noted below). Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).

- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)

- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.

- All the work on the exam should be your own. No "cooperation" is allowed.

- The exam has **7** problems and **12** pages. There are a total of **100** points.

Problem 1. Suppose (X, Y) has a joint density $f(x, y)$ given by

$$f(x, y) = \frac{1}{\pi \sigma \sqrt{1 - \rho^2}} \exp\left(-\frac{1}{2(1 - \rho^2)}\left[x^2 - 2\rho x\left(\frac{y - \mu}{\sigma}\right) + \left(\frac{y - \mu}{\sigma}\right)^2\right]\right)$$

$$\text{for } x \geq 0, \quad -\infty < y < \infty,$$

and $\quad f(x, y) = 0 \quad$ for $x < 0, \quad -\infty < y < \infty.$

The parameters satisfy $-1 < \rho < 1$, $\sigma > 0$, and $-\infty < \mu < \infty$.

(a) (12%) Find $f_X(x)$, the marginal density of X. (Make sure to specify $f_X(x)$ for all $x \in \mathbb{R}$.)

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[**Problem 1 continued**]

[Extra work space for Problem 1(a) if needed]

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[Extra work space for Problem 1(a) if needed]

(b) (12%) For $x > 0$ find $f_{Y|X}(y|x)$, the conditional density Y given $X = x$.

Problem 2.   Let $n > 0$ be some fixed integer. Suppose

$$X \mid P \sim \text{Binomial}(n, P)$$
$$P \sim \text{Beta}(\alpha, \beta).$$

(a)   (10%) Find $f_X(x)$, the marginal mass function of X.

(b) (6%) Find EX.

(c) (8%) Find $\text{Var}(X)$.

Problem 3. (16%) Suppose (X, Y) has joint density given by

$$f(x, y) = \frac{6x}{(1 + x + y)^4} \quad \text{for} \quad 0 < x < \infty, \ 0 < y < \infty.$$

Let $U = X/(X + Y)$ and $W = X + Y$. Find the joint density of (U, W). (Make sure to specify the support.)

Problem 4. (10%) Let X, Y, Z be uncorrelated random variables, each with mean μ and variance σ^2.

Find $\text{Cov}(5 + X + Y, X - Y - Z)$.

Problem 5. (8%) Let $X_0, X_1 X_2, \ldots$ be a Markov chain with initial distribution given by the row vector a and transition probability matrix P. Prove that $P(X_n = k) = (aP^n)_k$.

Problem 6. Suppose (X, Y) have the joint density

$$f(x, y) = \frac{2\exp(-\pi x)}{5 - 3\sin(y)} \quad 0 < x < \infty, \ 0 < y < 2\pi$$

where as usual $\pi = 3.141592653589793\ldots$

(a) (6%) Are X and Y independent? Answer "Yes" or "No" and then give a complete justification of your answer.

(b) (4%) Find $f_X(x)$, the marginal density of X. (State the answer and give a brief justification.)

No work is required for this problem.

Problem 7. Suppose you know $f_{W,X,Y,Z}(w, x, y, z)$, the joint density of the continuous random variables W, X, Y, Z. In terms of this, answer the following.

(a) (4%) Give an expression for $f_{X,Z}(x, z)$, the marginal density of (X, Z).

(b) (4%) Give an expression for $f_{W,Y|X,Z}(w, y|x, z)$, the conditional density of (W, Y) given (X, Z).