TEST #3
STA 5326 **Name:**____________________________________
December 10, 2019

Please read the following directions.
DO NOT TURN THE PAGE UNTIL INSTRUCTED TO DO SO

Directions

- The exam is closed book and closed notes. You will be supplied with scratch paper, and a copy of the Table of Common Distributions from the back of our textbook.
- During the exam, you may use ONLY what you need to write with (pens, pencils, erasers, etc) and (if you wish) an ordinary scientific calculator (TI-86 or below is fine).
- All other items (INCLUDING CELL PHONES) must be left at the front of the classroom during the exam. This includes backpacks, purses, books, notes, etc. You may keep small items (keys, coins, wallets, etc., but NOT CELL PHONEs) so long as they remain in your pockets at all times.
- Partial credit is available. (If you know part of a solution, write it down. If you know an approach to a problem, but cannot carry it out – write down this approach. If you know a useful result, write it down.)
- You must show and explain your work (including your calculations) for all the problems except those on the last two pages. **No credit is given without work or explanation!** But don't get carried away! Give enough explanation and work so that what you have done is clearly understandable.
- Make sure that the grader can easily see how you get from one step to the next. If you needed scratch paper to work something out, make sure to transfer your work to the exam.
- You should give only one answer to each problem. **Circle your answer** if there is any chance for confusion.
- Simplify your answers when it is easy to do so. But more difficult arithmetic does **not** have to be done completely (except as noted below). Answers can be left as fractions or products. You do not have to evaluate large binomial coefficients, factorials or powers. Answers can be left as summations (unless there is a simple closed form such as when summing a geometric or exponential series).
- All algebra and calculus must be done completely. (Only arithmetic can be left incomplete.)
- Do **not** quote homework results. If you wish to use a result from homework in a solution, you must prove this result.
- All the work on the exam should be your own. No "cooperation" is allowed.
- The exam has **7** problems and **12** pages. There are a total of **100** points.

General Remarks on grading:

I do NOT require students to simplify their arithmetic. They can leave in fractions, powers, factorials, etc. If you can see that their answer is correct, give them full credit. But students **must** *do all the necessary algebra and calculus to get full credit. They must compute all derivatives; they should lose points if they leave 'd/dx's or 'primes' in their answer. Also, student must simplify summations if there is a simple closed form.*

I usually do not deduct points for arithmetic errors unless they make the answer ridiculous. (For example, probabilities not between 0 and 1, negative expected values for positive-valued random variables, negative variances, etc.).

I usually do not deduct points for copying errors (unless they make the answer ridiculous).

I do deduct points for algebra and calculus errors, but as long as the student's solution is basically correct (except for the algebra or calculus mistakes), I try to give them at least half the credit.

However, if an algebra or calculus mistake is bad enough to transform the problem so that the solution after the point of the mistake no longer resembles the correct solution, then the student only gets credit for the work up to the point of the mistake.

In general, if a student makes a mistake which completely changes the problem, and then the student correctly solves the changed problem, they only get credit up to the point of the mistake; they don't get credit for correctly solving the wrong problem.

Problem 1. Suppose (X, Y) has a joint density $f(x, y)$ given by

$$f(x,y) = \frac{1}{\pi\sigma\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[x^2 - 2\rho x\left(\frac{y-\mu}{\sigma}\right) + \left(\frac{y-\mu}{\sigma}\right)^2\right]\right)$$

$$\text{for } x \geq 0, \quad -\infty < y < \infty,$$

$$\text{and} \quad f(x,y) = 0 \qquad \text{for } x < 0, \quad -\infty < y < \infty.$$

The parameters satisfy $-1 < \rho < 1$, $\sigma > 0$, and $-\infty < \mu < \infty$.

This is similar to exercise 4.45.

(a) (12%) Find $f_X(x)$, the marginal density of X. (Make sure to specify $f_X(x)$ for all $x \in \mathbb{R}$.)

The answer is $f_X(x) = \frac{2}{\sqrt{2\pi}} \exp(-x^2/2)$ *for* $x > 0$.

[Problem 1 continued]

[Extra work space for Problem 1(a) if needed]

(b) (12%) For $x > 0$ find $f_{Y|X}(y|x)$, the conditional density Y given $X = x$.

The answer is $f_{Y|X}(y|x) = \dfrac{1}{\sqrt{2\pi\sigma^2(1-\rho^2)}} \exp\left\{-\dfrac{(y-\mu-\rho\sigma x)^2}{2\sigma^2\sqrt{1-\rho^2}}\right\}$

Students should simplify their answers to the above form or something close to it.

Problem 2. Let $n > 0$ be some fixed integer. Suppose

$$\begin{aligned} X \mid P &\sim \text{Binomial}(n, P) \\ P &\sim \text{Beta}(\alpha, \beta). \end{aligned}$$

This is Example 4.4.6 and 4.4.8 and Exercise 4.34(a)

Students may leave Gamma functions in their answers.

(a) (10%) Find $f_X(x)$, the marginal mass function of X.

(b) (6%) Find EX.

(c) (8%) Find $\mathrm{Var}(X)$.

Problem 3. (16%) Suppose (X, Y) has joint density given by

$$f(x,y) = \frac{6x}{(1+x+y)^4} \quad \text{for} \quad 0 < x < \infty,\ 0 < y < \infty\,.$$

Let $U = X/(X+Y)$ and $W = X+Y$. Find the joint density of (U, W). (Make sure to specify the support.)

This is similar to exercise 4.19(b). See the homework solution.

Problem 4. (10%) Let X, Y, Z be uncorrelated random variables, each with mean μ and variance σ^2.

Find $\text{Cov}(5 + X + Y\ ,\ X - Y - Z)$.

This problem is similar to exercise 4.43.

Problem 5. (8%) Let $X_0, X_1 X_2, \ldots$ be a Markov chain with initial distribution given by the row vector a and transition probability matrix P. Prove that $P(X_n = k) = (aP^n)_k$.

The proof is on page 14 of notes13_markov_chains.pdf.

Students should NOT use $P_i(X_n = k) = p_{ik}^{(n)} \equiv (P^n)_{ik}$ *in their proofs since that is actually a special case of the result they are being asked to prove.*

Problem 6. Suppose (X, Y) have the joint density

$$f(x, y) = \frac{2\exp(-\pi x)}{5 - 3\sin(y)} \qquad 0 < x < \infty, \ 0 < y < 2\pi$$

where as usual $\pi = 3.141592653589793\ldots$

The easiest solution to this problem uses the Lemma stated (in various forms) on pages 10 and 11 of notes10.pdf.

(a) (6%) Are X and Y independent? Answer "Yes" or "No" and then give a complete justification of your answer.

The support of $f(x, y)$ is a Cartesian product set $A \times B$, and on this support the density has the form $cp(x)q(y)$. Therefore X and Y are independent.

Alternatively, students can show independence by verifying that the joint density is a product of the marginals. But this is more work since then they must find the marginals in some form.

(b) (4%) Find $f_X(x)$, the marginal density of X. (State the answer and give a brief justification.)

The Lemma also says that $f_X(x) = ap(x)$ for $x \in A$. Therefore, we have $f_X(x) = ae^{-\pi x}$ for $x > 0$. The normalizing constant must be $a = \pi$ so the answer is $f_X(x) = \pi e^{-\pi x}$ for $x > 0$.

Students must get the correct normalizing constant. *If the normalizing constant is wrong, they should lose 2 points (i.e., get only half credit).*

No work is required for this problem.

Problem 7. Suppose you know $f_{W,X,Y,Z}(w, x, y, z)$, the joint density of the continuous random variables W, X, Y, Z. In terms of this, answer the following.

See page 13 of notes12.pdf.

(a) (4%) Give an expression for $f_{X,Z}(x, z)$, the marginal density of (X, Z).

(b) (4%) Give an expression for $f_{W,Y|X,Z}(w, y|x, z)$, the conditional density of (W, Y) given (X, Z).